Auslander-Buchsbaum-Hochster

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Introduction

We try to present some of the results of the Chapter 5 and 6 of Northcott [1], in particular the Theorem of Auslander-Buchsbaum-Hochster (Theorem 2, Chapter 6), without using localisation at an arbitrary prime or minimal prime. The main (elementary) tool is the long exact sequence associated to a short exact sequence of complexes, in the particular case of the Koszul complex.

1 Koszul complex

1.1 Exterior algebra

Let R be an arbitrary commutative ring and M a R-module. The *exterior algebra* $\bigwedge(M)$ is the free algebra with a map $i : M \to \bigwedge(M)$ satisfying $i(a)i(a) = 0$ for all a in M. This implies $i(a)i(b) + i(b)i(a) = 0$ for all a, b in M.

In the case $M = R^n$, any element a of M is determined by a sequence $a_1, \ldots, a_n$ of elements in R. We can give a concrete realisation of $\bigwedge(M)$: we consider the free R-module V on the formal elements e_I, where I is a subset of $N_n = \{1, \ldots, n\}$. We write also $e_I = e_{i_1 \ldots i_k}$ whenever I is the set of elements $1 \leqslant i_1 < \ldots < i_k \leqslant n$. We denote by $|I|$ the cardinality of I. We define

$$e_I e_J = e_{I \cup J} \prod_{(i,j) \in I \times J} (i, j)$$

with $(i, j) = 1$ if $i < j$, and $(i, j) = -1$ if $j < i$ and $(i, j) = 0$ if $i = j$. This satisfies the associativity law $(e_I e_J)e_K = e_I(e_J e_K)$. We also have $e_i e_i = 0$ and $e_i e_j + e_j e_i = 0$. This multiplication extends canonically to V, with a map $i : R^n \to V$ defined by $i(a) = \Sigma a_j e_j$. We can verify that V together with the map $i : R \to V$ is a realisation of $\bigwedge(M)$ for $M = R^n$. It is natural to identify a and $i(a)$ for a in R^n.

In this case, $\bigwedge(M)$ is the direct sum of $n + 1$ free modules $\bigwedge^k(M)$ for $k = 0, \ldots, n$ where $\bigwedge^k(M)$ is generated by e_I for $|I| = k$. We write $\bigwedge^l(M) = 0$ for $n < l$. If a is in M and u is in $\bigwedge^k(M)$ then au is in $\bigwedge^{k+1}(M)$.

More generally if E is a module over R we define $C_l(E) = \oplus_{|I|=l} E e_I$. If u is in $\bigwedge^k(E)$ and $v = \Sigma v_I e_I$ is in $C_l(E)$ we define $uv = \Sigma v_I u e_I$ in $C_{k+l}(E)$.

1.2 Interior product

If a and b are in R^n we define $a \cdot b = \Sigma a_i b_i$. In particular $a \cdot e_i = a_i$. We define more generally by induction on k

$$a \cdot e_{i_0 \ldots i_k} = a_{i_0} e_{i_1 \ldots i_k} - e_{i_0}(a \cdot e_{i_1 \ldots i_k})$$

We then define $a \cdot v$ in $\bigwedge^k(R^n)$ for v in $\bigwedge^{k+1}(R^n)$ by linearity and we have

$$a \cdot (bu) = (a \cdot b)u - b(a \cdot u)$$

1.3 Complex

To any vector a in R^n we associate the complex $C_k(a; E) = C_k(E)$ and the derivation

$$d : C_k(E) \to C_{k+1}(E) \qquad v \longmapsto av$$

We have $d^2u = a(au) = aau = 0$. We write $H^i(a; E)$ the corresponding homology group. Any map $E \to F$ defines canonically a complex map $C_k(a; E) \to C_k(a; F)$ and if $E \to F \to G$ is exact then so is $C_k(a; E) \to C_k(a; F) \to C_k(a; G)$.

2 Grade

Let R be an arbitrary commutative ring, and a an element of R^n of coordinate $a_1, \dots, a_n$[1]. If E is a module over R we define $Gr(a; E) \geqslant k$ by $H^i(a; E) = 0$ for $i < k$. In particular:

1. $Gr(a; E) \geqslant 1$ means that $a_1, \dots, a_n$ is regular for E: if $a_i x = 0$ for all i then $x = 0$
2. $Gr(a; E) \geqslant 2$ iff $Gr(a; E) \geqslant 1$ and whenever we have a family x_i such that $a_i x_j - a_i x_j = 0$ then there exists x such that $x_i = a_i x$
3. $Gr(a; E) \geqslant 3$ iff $Gr(a; E) \geqslant 2$ and whenever we have a family x_{ij} such that $a_i x_{jk} - a_j x_{ik} + a_k x_{ij} = 0$ then there exists x_i such that $x_{ij} = a_i x_j - a_j x_i$
4. ...

Notice that we have $Gr(a; 0) \geqslant k$ for all k.

Lemma 2.1 *The multiplication by any element of $\langle a \rangle$ kills each $H^l(a; E)$.*

Proof. An element of $\langle a \rangle$ is of the form $b \cdot a$ for some b in R^n. We have

$$b \cdot (au) + a(b \cdot u) = (b \cdot a)u$$

which shows that $(b \cdot a)u$ is in the image of $v \longmapsto av$ if $au = 0$. □

We write $Gr(a) \geqslant k$ for $Gr(a; R) \geqslant k$.

3 Short exact sequence

Lemma 3.1 *If $0 \to E \to F \to G \to 0$ is a short exact sequence and $Gr(a; E) \geqslant k + 1$ and $Gr(a; F) \geqslant k$ then $Gr(a; G) \geqslant k$.*

Proof. We have a short exact sequence of complexes $0 \to C(a; E) \to C(a; F) \to C(a; G) \to 0$ to which we associate the long exact sequence

$$0 \to H^0(a; E) \to H^0(a; F) \to H^0(a; G) \to H^1(a; E) \to H^1(a; F) \to H^1(a; G) \to \dots$$

and so we have $H^i(a; G) = 0$ for all $i < k$ if $H^i(a; E) = 0$ for all $i \leqslant k$ and $H^i(a; F) = 0$ for all $i < k$. □

[1] The definition of $Gr(a; E) \geqslant k$ that follows will depend only on the ideal $\langle a \rangle = \langle a_1, \dots, a_n \rangle$.

This corresponds to Lemma 12, Chapter 5. We have the following reformulation of Lemma 13, Chapter 5, that has a similar proof.

Lemma 3.2 *If* $0 \to E \to F \to G \to 0$ *is a short exact sequence and* $Gr(a; F) \geqslant k+1$ *and* $Gr(a; G) \geqslant k$ *then* $Gr(a; E) \geqslant k+1$.

Proof. We have a short exact sequence of complexes $0 \to C(a; E) \to C(a; F) \to C(a; G) \to 0$ to which we associate the long exact sequence

$$0 \to H^0(a; E) \to H^0(a; F) \to H^0(a; G) \to H^1(a; E) \to H^1(a; F) \to H^1(a; G) \to \dots$$

and so we have $H^i(a; E) = 0$ for all $i \leqslant k$ if $H^i(a; F) = 0$ for all $i \leqslant k$ and $H^i(a; G) = 0$ for all $i < k$. $\square$

We get also a direct proof of Theorem 1, Chapter 6.

Theorem 3.3 *If* $0 \to R^n \xrightarrow{A} R^m \to E \to 0$ *is a short exact sequence and* $n > 0,\ m > 0$ *and all coefficients of* A *are in the ideal corresponding to* $\langle a \rangle$ *then* $Gr(a; E) \geqslant k$ *iff* $Gr(a) \geqslant k+1$.

Proof. Let L_i be $H^i(a; R)$. We associate to the given short exact sequence the long exact sequence

$$0 \to L_0^n \to L_0^m \to H^0(a; E) \to L_1^n \to L_1^m \to H^1(a; E) \to L_2^n \to L_2^m \to H^2(a; E) \to \dots$$

Each application $L_i^n \to L_i^m$ is represented by the matrix A, and hence it is 0 by Lemma 2.1. Since A represents an injective map and $n > 0,\ m > 0$ we have also $L_0 = 0$. It follows that $L_i = 0$ for all $i \leqslant k$ iff $H^i(a; E) = 0$ for all $i < k$. $\square$

4 Applications

Most applications are applications of Lemma 3.1. We start by Theorem 22, Chapter 5, which is due to Peskine and Szpiro.

Theorem 4.1 *If we have a complex* C

$$0 \to C_n \xrightarrow{d_n} \dots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

of R*-modules such that* $Gr(a; C_k) \geqslant k$ *and* $\langle a \rangle H_k(C) = 0$ *for* $k = 1, \dots, n$ *then* C *is exact.*

Proof. We write $B_k = \mathsf{Im}\ d_{k+1}$ and $C_k = Ker\ d_k$, so that $H_k = Z_k/B_k$, where we write H_k for $H_k(C)$.

Since $H_n = Z_n$ is a submodule of C_n and $Gr(a; C_n) \geqslant 1$ and $\langle a \rangle H_n = 0$ we have $H_n = 0$ and hence $0 \to C_n \to C_{n-1}$ is exact. Furthermore, $B_{n-1} = \mathsf{Im}\ d_n$ is isomorphic to H_n and so $Gr(a; B_{n-1}) \geqslant n$.

Assume $n \geqslant 2$. Since Z_{n-1} is a submodule of C_{n-1} we have $Gr(a; Z_{n-1}) \geqslant n-1 \geqslant 1$. Lemma 3.1 and the short exact sequence

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1} \to 0$$

shows then that $Gr(a; H_{n-1}) \geqslant 1$ and hence $H_{n-1} = 0$ since $\langle a \rangle H_{n-1} = 0$. Using Lemma 3.1 again on the short exact sequence

$$0 \to B_{n-1} \to C_{n-1} \to B_{n-2} \to 0$$

we have $Gr(a; B_{n-2}) \geqslant n-1$. If $n \geqslant 3$ we then deduce similarly $H_{n-2} = 0$ and $Gr(a; B_{n-3}) \geqslant n-2,\ \dots$ $\square$

We decompose Theorem 2, Chapter 6 (Auslander-Buchsbaum-Hochster's Theorem) in two parts.

Theorem 4.2 *If we have a finite free resolution of a module E*

$$0 \to F_m \to \ldots \to F_0 \to E \to 0$$

with $F_i = R^{n_i}$ and $Gr(a) \geqslant k + m$ then $Gr(a; E) \geqslant k$.

Proof. We have the exact sequences

$$0 \to E_1 \to F_0 \to E \to 0 \quad 0 \to E_2 \to F_1 \to E_1 \to 0 \ \ldots \quad 0 \to F_m \to F_{m-1} \to E_{m-1} \to 0$$

and so, using Lemma 3.1, we get successively $Gr(a; E_i) \geqslant k + i$ for $i = m - 1, m - 2, \ldots$ until we have $Gr(a; E) \geqslant k$. $\square$

The following Corollary corresponds to Theorem 4, Chapter 6, which is proved in Northcott via localisation at a minimal prime ideal.

Corollary 4.3 *If we have a finite free resolution of a module E*

$$0 \to F_m \to \ldots \to F_0 \to E \to 0$$

with $F_i = R^{n_i}$ and $Gr(a) \geqslant m + 1$ and $\langle a \rangle E = 0$ then $E = 0$.

Proof. We have $Gr(a; E) \geqslant 1$ by Theorem 4.2 and so $\langle a \rangle E = 0$ implies $E = 0$. $\square$

Another Corollary corresponds to the Exercise 7, Chapter 7.

Corollary 4.4 *If we have a $n \times (n + 1)$ matrix A such that $Gr(\Delta_n(A)) \geqslant 3$ then $1 = \Delta_n(A)$.*

Proof. We use the fact that if $Gr(\Delta_n(A)) \geqslant 2$ then we have an exact sequence

$$0 \to R^n \xrightarrow{A} R^{n+1} \to R \to R/\Delta_n(A) \to 0$$

(this is proved in HilbertBurchPart2). The claim follows then from Corollary 4.3. $\square$

Yet another Corollary corresponds to one direction of Theorem 15, Chapter 6.

Corollary 4.5 *We assume given a complex*

$$(*) \qquad 0 \to F_m \to \ldots \to F_0$$

with $F_i = R^{n_i}$. We define $r_m = n_m$ and $r_{k-1} = n_{k-1} - r_k$ for $k = m, \ldots, 1$ assuming that we have $r_k \leqslant n_{k-1}$ at each step. Let the map $F_l \to F_{l-1}$ be represented by the matrix A_l and $I_l = \Delta_{r_l}(A_l)$. If $Gr(I_l) \geqslant l$ for $l = m, \ldots, 1$ then the complex $(*)$ is exact.

Proof. The result is clear if $m = 1$ since $Gr(\Delta_{n_1}(A_1)) \geqslant 1$ implies that the map represented by A_1 is injective. The proof is then by induction on m. By induction, we get a finite free resolution of $F_1/\mathsf{Im}\ A_2$

$$0 \to F_m \to \ldots F_1 \to F_1/\mathsf{Im}\ A_2 \to 0$$

It follows then from Theorem 4.2 that we have $Gr(I_m; F_1/\mathsf{Im}\ A_2) \geqslant 1$. This implies that, for showing that $F_2 \to F_1 \to F_0$ is exact, it is enough to show it after localisation at each element of I_m: if x is in F_1 and $A_1\ x = 0$, the x is 0 in $F_1/\mathsf{Im}\ A_2$ if it is 0 after localisation at each element of I_m since $Gr(I_m; F_1/\mathsf{Im}\ A_2) \geqslant 1$. But if we localize at an element of I_m we can find a finite free module E_{m-1} such that $F_{m-1} = E_{m-1} \oplus \mathsf{Im}\ A_m$, and the complex

$$E_{m-1} \to \ldots \to F_1 \to F_0$$

is exact by induction. $\square$

When the complex $(*)$ is exact, one can show that we have also $\Delta_{r_k+1}(A_k) = 0$. Thus the matrix A_k is stable of rank r_k: the ideal $\Delta_{r_k}(A_k)$ is regular and $\Delta_{r_k+1}(A_k)$ is 0.

We finally come to the other direction for the Auslander-Buchsbaum-Hochster's Theorem.

Theorem 4.6 *If we have a finite free resolution of a module E*

$$0 \to F_m \to \ldots \to F_0 \to E \to 0$$

with $F_i = R^{n_i}$ and $n_i > 0$ and the matrix A_m representing the map $F_m \to F_{m-1}$ has its coefficients in $\langle a \rangle$ and $Gr(a; E) \geqslant k$ then $Gr(a) \geqslant k + m$.

Proof. We have the exact sequences

$$0 \to E_1 \to F_0 \to E \to 0 \quad 0 \to E_2 \to F_1 \to E_1 \to 0 \ \ldots \quad 0 \to F_m \to F_{m-1} \to E_{m-1} \to 0$$

Since A_m represents an injective map and has its coefficients in $\langle a \rangle$ we have $Gr(a) \geqslant 1$ and hence $Gr(a; F_i) \geqslant 1$ for all i. It follows that we have $Gr(a; E_i) \geqslant 1$ for $i = 1, \ldots, m-1$. Using then Theorem 3.3, we deduce $Gr(a) \geqslant 2$. We can then use Lemma 3.2 to deduce $Gr(a; E_i) \geqslant 2$ for $i = 2, \ldots, m-1$. We can then use Theorem 3.3 to deduce $Gr(a) \geqslant 3$, $\ldots$ until we get $Gr(a) \geqslant k + m$. $\square$

References

[1] D.G. Northcott. Finite Free Resolutions. Cambridge Tracts in Mathematics, 1976.