Regular entailment relations

Introduction

If G is an ordered commutative group and we have a map $f : G \to L$ where L is a l-group, we can define a relation $A \vdash B$ between *non empty* finite sets of G by $\wedge f(A) \leqslant \vee f(B)$. This relation satisfies the conditions

1. $a \vdash b$ if $a \leqslant b$ in G
2. $A \vdash B$ if $A \supseteq A'$ and $B \supseteq B'$ and $A' \vdash B'$
3. $A \vdash B$ if $A, x \vdash B$ and $A \vdash B, x$
4. $A \vdash B$ if $A + x \vdash B + x$
5. $a + x, b + y \vdash a + b, x + y$

We call a *regular entailment relation* on an ordered group G any relation which satisfies these conditions. The remarkable last condition is called the *regularity condition*. Note that the converse relation of a regular entailment relation is a regular entailment relation.

Any relation satisfying the three first conditions define in a canonical way a (non bounded) distributive lattice L. The goal of this note is to show that this distributive lattice has a (canonical) l-group structure.

1 General properties

A first consequence of regularity is the following.

Proposition 1.1 *We have $a, b \vdash a + x, b - x$ and $a + x, b - x \vdash a, b$. In particular, $a \vdash a + x, a - x$ and $a + x, a - x \vdash a$*

Proof. By regularity we have $(-x + a + x), (b + 2x - 2x) \vdash (-x + b + 2x), (a + x - 2x)$. The other claim is symmetric. □

Corollary 1.2 $\wedge A \leqslant (\wedge A + x) \vee (\wedge A - x)$.

Proof. We can reason in the distributive lattice L defined by the given (non bounded) entailment relation and use Proposition 7.3. □

Corollary 1.3 *If we have $A, A + x \vdash B$ and $A, A - x \vdash B$ then $A \vdash B$. Dually, if $A \vdash B, B + x$ and $A \vdash B, B - x$ then $A \vdash B$.*

Lemma 1.4 *We have $A, A + x \vdash B$ iff $A \vdash B, B - x$*

Proof. We assume $A, A + x \vdash B$ and we prove $A \vdash B, B - x$. By Corollary 1.3, it is enough to show $A, A - x \vdash B, B - x$ but this follows from $A, A + x \vdash B$ by translation by $-x$ and then weakening. The other direction is symmetric. □

Lemma 1.5 *If $0 \leqslant p \leqslant q$ then $a, a + qx \vdash a + px$*

Proof. We prove this by induction on q. This holds for $q = 0$. If it holds for q, we note that we have $a, a+(q+1)x \vdash a+x, a+qx$ by regularity and since $a, a+qx \vdash a+x$ by induction we get $a, a+(q+1)x \vdash a+x$ by cut. By induction we have $a, a + qx \vdash a + px$ for $p \leqslant q$ and hence $a + x, a + (q + 1)x \vdash a + (p + 1)x$. By cut with $a, a + (q + 1)x \vdash a + x$ we get $a, a + (q + 1)x \vdash a + (p + 1)x$. □

Given a regular entailment relation $\vdash$ and an element x, we describe now the *regular* entailment relation $\vdash_x$ where we force $0 \vdash_x x$.

We define by $A \vdash_x B$ iff there exists p such that $A, A + px \vdash B$ iff (by Lemma 1.4) there exists p such that $A \vdash B, B - px$, and we are going to show that this is the least regular entailment relation containing $\vdash$ and such that $0 \vdash_x x$. We have $0 \vdash_x x$ since $0, x \vdash x$.

Note that, by using Lemma 1.5, if we have $A, A + px \vdash B$, we also have $A, A + qx \vdash B$ for $q \geqslant p$.

Proposition 1.6 *The relation $\vdash_x$ is a regular entailment relation.*

Proof. The only complex case is the cut rule. We assume $A, A + px \vdash B, u$ and $A, A + qx, u, u + qx \vdash B$ and we prove $A \vdash_x B$. By Lemma 1.5, we can assume $p = q$. We write $y = px$ and we have $A, A+y \vdash B, u$ and $A, A + y, u, u + y \vdash B$. We write $C = A, A + y, A + 2y$ and we prove $C \vdash B$.

We have by weakening $C \vdash B, u$ and $C, u, u+y \vdash B$ and $C \vdash B+y, u+y$. By cut, we get $C, u \vdash B, B+y$. By Lemma 1.4, this is equivalent to $C, u, C-y, u-y \vdash B$. We also have $C, u, C+y, u+y \vdash B$ by weakening from $C, u, u + y \vdash B$. Hence by Lemma 1.3 we get $C, u \vdash B$. Since we also have $C \vdash B, u$ we get $C \vdash B$ by cut.

By Lemma 1.5 we have $A, A + 2y \vdash B$, which shows $A \vdash_x B$. □

Proposition 1.7 *If $A \vdash_x B$ and $A \vdash_{-x} B$ then $A \vdash B$*

Proof. We have $A, A + px \vdash B$ and $A, A - qx \vdash B$. Using Lemma 1.5 we can assume $p = q$ and then conclude by Lemma 7.3. □

Proposition 1.7 implies that to prove an entailment involving some elements, we can always assume that these element are linearly ordered for the relation $a \vdash b$. Here are two direct applications.

Proposition 1.8 *We have $A \vdash b_1, \dots, b_m$ iff $A - b_1, \dots, A - b_m \vdash 0$.*

Thus $A \vdash B$ iff $A - B \vdash 0$ iff $0 \vdash B - A$.

Proposition 1.9 *If $A + b_1, \dots, A + b_m \vdash b_j$ for $j = 1, \dots, m$ then $A \vdash 0$.*

It follows from Proposition 1.9 that if we consider the monoid of formal elements $\wedge A$, with the operation $\wedge A + \wedge B = \wedge(A + B)$, ordered by the relation $\wedge A \leqslant \wedge B$ iff $A \vdash b$ for all b in B, we have a *cancellative* monoid.

It follows then from Proposition 1.8 that the distributive lattice defined by the Grothendieck l-group associated to this cancellative monoid coincides with the distributive lattice defined by the relation $\vdash$.

Here is another consequence of the fact that we can always assume that these element are linearly ordered for the relation $a \vdash b$.

Corollary 1.10 *If $a_1 + \dots + a_n = 0$ then $a_1, \dots, a_n \vdash 0$.*

Corollary 1.11 *If $a_1 + \dots + a_n = b_1 + \dots + b_n$ then $a_1, \dots, a_n \vdash b_1, \dots, b_n$.*

Proof. We have $\Sigma_{i,j} a_i - b_j = 0$ and we can apply the previous result. □

2 Another presentation of regular entailment relations

It follows from Proposition 1.8 that the relation $\vdash$ is completely determined by the predicate $A \vdash 0$ on non empty finite subsets of the group. Let us analyse what are the properties satisfied by this predicate $R(A) = A \vdash 0$. First, is satisfies

(R_1) $R(a)$ whenever $a \leqslant 0$ in G.

Then, it is *monotone*

(R_2) $R(A)$ holds if $R(A')$ and $A' \subseteq A$

The cut-rule can be stated as $R(A - B)$ if $R(A - B, x - B)$ and $R(A - B, A - x)$, so we get the property (since we can assume $x = 0$ by translation and replace B by $-B$)

(R_3) $R(A + B)$ if $R(A + B, A)$ and $R(A + B, B)$

Finally, the regularity condition gives $R(a - b, b - a, x - y, y - x)$ which simplifies using (R_2) to

(R_4) $R(x, -x)$

We get in this way another presentation of a regular entailment relation as a predicate satisfying the conditions $(R_1), (R_2), (R_3), (R_4)$. If R satisfies these properties and we define $A \vdash B$ by $R(A - B)$ then we get a regular entailment relation. (We have one less axiom since the translation property $A \vdash B$ if $A + x \vdash B + x$ is automatically satisfied.)

3 System of ideals

Let us use the same analysis for the notion of *system of ideals*, which is a relation $A \vdash x$ between non empty finite subsets of G and element x in G satisfying the conditions

1. $a \vdash x$ if $a \leqslant x$ in G
2. $A \vdash x$ if $A \supseteq A'$ and $A' \vdash x$
3. $A \vdash x$ if $A, y \vdash x$ and $A \vdash y$
4. $A \vdash x$ if $A + y \vdash x + y$

We consider the predicate $S(A) = A \vdash 0$. This predicate satisfies

(R_1) $S(a)$ if $a \leqslant 0$ in G

(R_2) $S(A)$ holds if $A \supseteq A'$ and $S(A')$

(R_5) $S(A)$ holds if $S(A, u)$ and $S(A - u)$

Conversely if S satisfies $(R_1), (R_2)$ and (R_5) and we define $A \vdash_S x$ by $S(A - x)$ then $\vdash$ is a system of ideals.

Clearly to a system of ideals we can associate the relation $A \leqslant_S B$ by $A \vdash_S b$ for all b in B we define a preordered monoid, with $A + B$ as monoid operation. Conversely, any preorder on the monoid of finite non empty subset with $A \wedge B$ being A, B as meet operation and $A + B$ as monoid operation, defines a system of ideal $A \vdash b = \wedge A \leqslant b$.

4 Regularisation of a system of ideals

Note that both notions (reformulation of regular relations) and system of ideals are now predicates on nonempty finite subsets of G. We say that a system of ideals is *regular* if it satisfies (R_3) and (R_4).

The following proposition follows from Proposition 1.9.

Proposition 4.1 *The preordered monoid $\leqslant_S$ is cancellative if, and only if, S is regular.*

Proof. If S is regular then $\leqslant_S$ is cancellative by Proposition 1.9. Conversely, if $\leqslant_S$ is conservative then the meet-semilattice it defines embeds in its Grothendieck group, which is a distributive lattice. □

We always have the *least* system of ideals: $S_0(A)$ iff A contains an element $\leqslant 0$ in G. This clearly satisfies (R_1) and (R_2) and it satisfies (R_5): if $S_0(A, u)$ then either $S_0(A)$ or $u \leqslant 0$ and if $u \leqslant 0$ then $S_0(A - u)$ implies $S_0(A)$.

Note also that systems of ideal are closed by arbitrary intersection and directed union.

Let S be a system of ideals. We define $T_x(S)$ to be the least system of ideals Q containing S and such that $Q(x)$. We have $T_xT_y = T_yT_x$ and $T_x(S \cap S') = T_x(S) \cap T_x(S')$ directly from this definition. Lorenzen found an elegant direct description of $T_x(S)$.

Proposition 4.2 *$T_x(S)(A)$ iff there exists $k \geqslant 0$ such that $S(A, A - x, \ldots, A - kx)$.*

Proof. If we have $A, A - x, \ldots, A - kx \leqslant_S u$ and $A, A - x, \ldots, A - lx, u, u - x, \ldots, u - lx \leqslant_S v$ then we have by l cuts $A, A - x, \ldots, A - (k + l)x \leqslant_S v$. □

Note that it does not seem that we can simplify this condition to $S(A, A - kx)$ in general.

We next define $U_x(S) = T_x(S) \cap T_{-x}(S)$. We have $U_xU_y = U_yU_x$.

Lemma 4.3 *If S is a system of ideals such that $U_x(S) = S$ for all x then S is regular.*

Proof. We show that conditions (R_3) and (R_4) hold.

We have $S(x, -x)$ since we have both $T_x(S)(x, -x)$ and $T_{-x}(S)(x, -x)$. This shows (R_3).

Let us show (R_4). We assume $\wedge(A + B) \wedge \wedge B \leqslant_S 0$ and $\wedge(A + B) \wedge \wedge A \leqslant_S 0$ and we show $\wedge(A + B) \leqslant_S 0$.

Note that we have $T_a(S)(A + B)$ for any a in A by monotonicity: if we force $a \leqslant_S 0$ then $\wedge(A + B) \leqslant_{T_a(S)} \wedge B$ and so $\wedge(A+B) \leqslant_{T_a(S)} 0$ follows from $\wedge(A+B) \wedge \wedge B \leqslant_{T_a(S)} 0$. Let T be the composition of all T_{-a} for a in A; we force $0 \leqslant_S a$ for all a in A. We have $\wedge B \leqslant_{T(S)} \wedge(A + B)$ and so $\wedge B \leqslant_{T(S)} 0$ follows from $\wedge(A + B) \wedge \wedge B \leqslant_{T(S)} 0$. This implies $\wedge(A + B) \leqslant_{T(S)} \wedge A$ and so $\wedge(A + B) \leqslant_{T(S)} 0$ follows from $\wedge(A + B) \wedge \wedge A \leqslant_{T(S)} 0$.

We have $\wedge(A + B) \leqslant_{T_a(S)} 0$ for all a in A and $\wedge(A + B) \leqslant_{T(S)} 0$, so we get $\wedge(A + B) \leqslant 0$ as desired. □

It follows that if we define $L(S)$ to be the (directed) union of all $U_{x_1} \ldots U_{x_n}(S)$ we have that $L(S)$ is the least regular system containing S, so is the *regular closure* of S.

5 Constructive version of Lorenzen-Dieudonné Theorem

In particular, we can start from the least system of ideal. In this case, we have $L(S_0)(A)$ iff there exists $x_1, \ldots, x_n$ such that for any choice $\epsilon_1, \ldots, \epsilon_n$ for $-1, 1$ we can find $k_1, \ldots, k_n \geqslant 0$ and a in A such that $a + \Sigma\epsilon_ik_ix_i \leqslant 0$. We clearly have by elimination: if $L(S_0)(a)$ then $na \leqslant 0$ for some $n > 0$. We can then deduce from this a constructive version of Lorenzen-Dieudonné Theorem.

Theorem 5.1 *For any commutative ordered group G we can build a l-group L and a map $f : G \to L$ such that $f(a) \geqslant 0$ iff there exists $n > 0$ such that $na \geqslant 0$. More generally, we have $f(a_1) \wedge \cdots \wedge f(a_k) \geqslant 0$ iff there exists $n_1, \ldots, n_k \geqslant 0$ such that $\Sigma n_ia_i \geqslant 0$ and and $\Sigma n_i > 0$.*

6 Prüfer's definition of the regular closure

Prüfer found the following direct definition of the regular closure P, which follows directly from Proposition 4.1.

Theorem 6.1 *The regular closure R of a system of ideals S can be defined by $R(A)$ iff there exists B such that $A + B \leqslant_S B$.*

This gives another proof that if we have $L(S_0)(a)$ then $na \leqslant 0$ for some $n > 0$: if we have B such that $a + B \leqslant_{S_0} B$ then we have a cycle $a + b_2 \leqslant b_1, \ldots, a + b_1 \leqslant b_n$ and then $na \leqslant 0$.

7 Non commutative version

If G is an ordered group non necessarily commutative, we use a multiplicative notation and we define a *regular entailment relation* by the conditions

1. $a \vdash b$ if $a \leqslant b$ in G
2. $A \vdash B$ if $A \supseteq A'$ and $B \supseteq B'$ and $A' \vdash B'$
3. $A \vdash B$ if $A, x \vdash B$ and $A \vdash B, x$
4. $A \vdash B$ if $xAy \vdash xBy$
5. $xa, by \vdash xb, ay$

If $\vdash$ is a regular entailment relation, and V is the corresponding distributive lattice then we have a left and right action of G on V.

We define $\leqslant_{a,b}$ to be the least lattice quotient on V with left and right action of G such that $b \leqslant_{a,b} a$.

We define $u \leqslant^{a,b} v$ by $xa \wedge uy \leqslant xb \vee vy$ for all x and y in G.

Lemma 7.1 *We have $xa \wedge by \leqslant xb \vee ay$ for all a and b in V and all x and y in G.*

Proof. For instance, if we have $xa_1 \wedge by \leqslant xb \vee a_1 y$ and $xa_1 \wedge by \leqslant xb \vee a_2 y$ then we get $xa \wedge by \leqslant xb \vee ay$ for $a = a_1 \wedge a_2$ and for $a = a_1 \vee a_2$. □

Proposition 7.2 *$\leqslant^{a,b}$ defines a lattice quotient on V with left and right action of G on V such that $b \leqslant^{a,b} a$ if a and b are in G.*

Proof. We have $b \leqslant^{a,b} a$ since $xa \wedge by \leqslant xb \vee ay$ for all x and y by the previous Lemma.

If we have $u \leqslant^{a,b} v$ and $v \leqslant^{a,b} w$ then $xa \wedge uy \leqslant xb \vee vy$ and $xa \wedge vy \leqslant xb \vee wy$ for all x and y By cut, we get $xa \wedge uy \leqslant xb \vee wy$ for all x and y that is $u \leqslant^{a,b} w$. This shows that the relation $\leqslant^{a,b}$ is transitive. This relation is also reflexive since $xa \wedge uy \leqslant xb \vee uy$ for all x and y in G.

Finally, if we have $u \leqslant^{a,b} v$ that is $xa \wedge uy \leqslant xb \vee vy$ for all x and y in G, we also have $zut \leqslant^{a,b} zvt$ that is $xa \wedge zuty \leqslant xb \vee zvty$ for all x and y in G since we have $z^{-1}xa \wedge uty \leqslant z^{-1}xb \vee vty$ for all x and y in G. □

By definition $u \leqslant_{a,b} v$ implies $u \leqslant^{a,b} v$ since $\leqslant_{a,b}$ is the *least* invariant order relation forcing $a \leqslant_{a,b} b$.

Also by definition, note that we have $u \leqslant^{a,b} v$ iff $a \leqslant^{u,v} b$.

Proposition 7.3 *$u \leqslant_{a,b} v$ and $u \leqslant_{b,a} v$ imply $u \leqslant v$.*

Proof. Indeed $u \leqslant_{a,b} v$ implies $u \leqslant^{a,b} v$ which implies $a \leqslant^{u,v} b$. Together with $u \leqslant_{b,a} v$ this implies $u \leqslant^{u,v} v$ and so $xu \wedge uy \leqslant xv \vee vy$ for all x, y. In particular, for $x = y = 1$ we have $u \leqslant v$. □

It follows from this that V admits a group structure which extends the one on G. Indeed, Proposition 7.3 reduces the verification of the required equations to the case where G is totally ordered and $V = G$ in this case.