Inductively generated formal topologies

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Abstract

Formal topology aims at developing general topology in intuitionistic and predicative mathematics. Many classical results of general topology have been already brought into the realm of constructive mathematics by using formal topology and also new light on basic topological notions was gained with this approach which allows distinction which are not sensible in classical topology.

Here we give a systematic exposition of one of the main tools in formal topology: inductive generation. In fact, many formal topologies can be presented in a predicative way by an inductive generation and thus their properties can be proved inductively. We show however that some natural complete Heyting algebra cannot be inductively defined.

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Foreword

The aim of formal topology is to develop topology in a constructive framework where "constructive" is meant here to include both intuitionistic and predicative. We can fix, if desidered, such a foundational theory to be Martin-Löf's constructive set theory [ML84], but we actually often do not use its full strength. A monograph on formal topology is under preparation; it will include all the preliminaries on type theory which we are here compelled to skip, and more details on the basic notions.

Also other approaches to intuitionistic topology have been developed, notably the theory of locales, which is usually developed in topos theory. Working, as we do, with a foundational theory without power-sets brings to distinctions between notions or methods which are irrelevant in a foundation like topos theory, and hence there neglected. The most striking example is that it does not seem possible to define predicatively the (co)product of two formal topologies, unless they are inductively generated. It also brings us to new, sometime unexpected connections, like that with inductive definitions, which constitute a key tool for proof theory (cfr. [BFPS81] and [Acz77]).

The paper is organized as follows. We first give a short introduction to formal topology by showing how to move from the classical and impredicative case of a concrete topological space to the constructive and abstract notion of formal topology. Then the main topic of the paper is discussed, that is, the problem of inductive generation of formal topologies. We will both show the problems that must be solved to inductively generate a formal topology and we will present our solutions of such problems. Finally, the last part of the paper is devoted to show that most of the interesting formal topologies can be generated inductively. But we will also show that there are formal topologies that cannot be inductively generated; noticeably, we will also show that the example of non-inductively generated formal topology that we were able to build is classically equivalent to a formal topology which *can* be inductively generated.

This paper has a long history. Some of the relevant sources can be found in [FG82, Sam87, Coq92, NV97, CN96, Dra87].

In the whole paper we follow the notation introduced in [SV98]. We are confident that the reader will understand the notation with no problem since most of the work in [SV98] was made purposely to be able to use standard mathematical notation and still remain completely within Martin-Löf constructive set theory. Anyhow it can be useful to stress at least one thing: we will use the standard symbol $\in$ to mean the membership relation between an element and a *set* or a *collection*, while we switch to the symbol ε for the relation of membership between an element and a *subset*, since we want to stress the fact that a subset is never a set but just a propositional function. It can be useful to recall that, provided S is a set, a one of its elements and U one of its subsets, $a \in S$ is a judgement while $a\varepsilon U$ is a proposition such that $a\varepsilon U$ is true if and only if $U(a)$ is true; hence two subsets U and V of S are extensionally equal, notation $U =^S V$, if and only if $(\forall x \in S)\ (U(x) \leftrightarrow V(x))$.

1 The notion of formal topology

We recall in this section some of the motivations which lead to the notion of formal topology, which is the central tool of the approach to constructive topology adopted here.

It is convenient to start from a short analysis of the traditional definition of topological space, so that we can underline which steps are problematic from a constructive point of view, and how they are solved.

1.1 Concrete topological spaces

The classical definition reads: $(X, \Omega(X))$ is a topological space if X is a set and $\Omega(X)$ is a subset of $\mathcal{P}(X)$ which satisfies:

(Ω_1) $\emptyset, X \in \Omega(X)$;

(Ω_2) $\Omega(X)$ is closed under finite intersection;

(Ω_3) $\Omega(X)$ is closed under arbitrary union.

Usually, elements of X are called points and elements of $\Omega(X)$ are called opens.

The quantification implicitly used in (Ω_3) is of the *third* order, since it says $(\forall F \subseteq \Omega(X)) \cup F \in \Omega(X)$, that is $(\forall F \in \mathcal{P}(\mathcal{P}(X)))(F \subseteq \Omega(X) \to \cup F \in \Omega(X))$. The idea is that we can "go down" one step by thinking of $\Omega(X)$ as a family of subsets indexed by a set S through a map $\mathsf{N} : S \to \mathcal{P}(X)$, since we can now quantify over S rather than on $\Omega(X)$.

But we still have to say $(\forall U \in \mathcal{P}(S))(\exists c \in S)\ (\cup_{a\varepsilon U} \mathsf{N}(a) = \mathsf{N}(c))$, which is also impredicative.

We can "go down" another step by defining opens to be of the form $\mathsf{N}(U) \equiv \cup_{a\varepsilon U} \mathsf{N}(a)$ for an arbitrary subset U of S. In this way $\emptyset$ is open, because $\mathsf{N}(\emptyset) = \emptyset$, and closure under union is automatic, because obviously $\cup_{i\in I} \mathsf{N}(U_i) = \mathsf{N}(\cup_{i\in I} U_i)$. So, all we have to do is to require $\mathsf{N}(S)$ to be the whole X and closure under finite intersections, that is, condition (Ω_2). It is not difficult to realize that this amounts to the standard definition saying that $\{\mathsf{N}(a) \subseteq X |\ a \in S\}$ is a base (see for instance [Eng77]). So we reach the following definition:

Definition 1.1 *A* concrete topological space *is a triple* $\mathcal{X} \equiv (X, S, \mathsf{N})$ *where* X *is a set of* concrete points*,* S *is a set of* observables*,* N *is a map from* S *into* *subsets of X, called the* neighborhood *map, which satisfies*

$$(B_1) \quad X = \cup_{a \in S} \mathsf{N}(a)$$

$$(B_2) \quad (\forall a, b \in S)(\forall x \in X)\ (x \varepsilon \mathsf{N}(a) \cap \mathsf{N}(b) \rightarrow \\ (\exists c \in S)\ (x \varepsilon \mathsf{N}(c)\ \&\ \mathsf{N}(c) \subseteq \mathsf{N}(a) \cap \mathsf{N}(b)))$$

Note that this definition re-establishes a balance between the side of points, which we call the *concrete side*, and the side of observables, or formal basic neighbourhoods, which we call the *formal side*.

Note that (B_2) is just a rigorous writing of the usual condition stating that if $\mathsf{N}(a)$ and $\mathsf{N}(b)$ are two neighbourhoods of x then there exists a neighborhood $\mathsf{N}(c)$ of x which is contained both in $\mathsf{N}(a)$ and $\mathsf{N}(b)$ and this is all what we need to obtain closure under intersection.

Now, a map $\mathsf{N} : S \rightarrow \mathcal{P}(X)$ is a propositional function with two arguments, that is $\mathsf{N}(x)(a)\ \mathsf{prop}\ [x : X, a : S]$, that is a binary relation. Then we write it more suggestively as

$$x \Vdash a\ \mathsf{prop}\ [x : X, a : S]$$

and read it "x lies in a" or "x forces a".

It is convenient to use also a few abbreviations:

$$\begin{array}{rcl} x \Vdash U & \equiv & (\exists b \varepsilon U)\ x \Vdash b \\ \mathsf{ext}(a) & \equiv & \{x : X \mid x \Vdash a\} \\ \mathsf{ext}(U) & \equiv & \cup_{a \varepsilon U} \mathsf{ext}(a) \end{array}$$

Hence $x \Vdash a$ is the same as $x \varepsilon \mathsf{ext}(a)$ and $x \Vdash U$ is the same as $x \varepsilon \mathsf{ext}(U)$; thus the map N coincides with ext.

Then (B_1) and (B_2) can be rewritten as

$$(B_1) \quad (\forall x \in X)(\exists a \in S)\ x \Vdash a$$

$$(B_2) \quad (\forall a, b \in S)(\forall x \in X)\ ((x \Vdash a)\ \&\ (x \Vdash b) \rightarrow \\ (\exists c \in S)\ (x \Vdash c\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)))$$

We can make (B_2) a bit shorter by introducing another abbreviation, that is

$$a \downarrow b \equiv \{c : S \mid \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)\}$$

and by writing $x \varepsilon \mathsf{ext}(a) \cap \mathsf{ext}(b)$ for $x \Vdash a\ \&\ x \Vdash b$, so that it becomes

$$(B_2) \quad (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{ext}(a \downarrow b)$$

which looks much better.

Note that $c \varepsilon a \downarrow b$ implies that $\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$, so that $\mathsf{ext}(a \downarrow b) \equiv \cup_{c \varepsilon a \downarrow b} \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$. Then the definition of concrete topological space can be rewritten as follows:

Definition 1.2 *A* concrete topological space *is a triple $\mathcal{X} \equiv (X, S, \Vdash)$ where X and S are sets and $\Vdash$ is a binary relation from X to S satisfying:*

$$(B_1) \quad (\forall x \in X)\ x \Vdash S$$

$$(B_2) \quad (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) = \mathsf{ext}(a \downarrow b)$$

1.2 Formal topologies

The notion of formal topology arises by describing as well as possible the structure induced by a concrete topological space on the formal side and then by taking the result as an axiomatic definition. The reason for such a move is that the definition of concrete topological space is too restrictive, given that in the most interesting cases of topological space we do not have, from a constructive point of view, a *set* of points to start with. Thus, we choose two primitives, that is $\lhd$ and Pos[1], whose definition in the concrete case is

$$a \lhd U \equiv (\forall x \in X)\ (x \Vdash a \to x \Vdash U)$$

$$\mathsf{Pos}(a) \equiv (\exists x \in X)\ x \Vdash a$$

and look for their properties which are expressible without mentioning X and its elements.

Given that $x \Vdash U \equiv (\exists b \varepsilon U)\ x \Vdash b$, the rule of $\exists$-introduction yields that if $a \varepsilon U$ then $a \lhd U$. Similarly, since $(\forall x \in X)\ (x \Vdash U \to x \Vdash V)$ is logically equivalent to $(\forall b \varepsilon U)(\forall x \in X)\ (x \Vdash b \to x \Vdash V)$, the rule of $\exists$-elimination yields that if $a \lhd U$ and $U \lhd V$ then $a \lhd V$, where $U \lhd V$ is a shorthand for $(\forall b \varepsilon U)\ b \lhd V$ which we will use from now on.

Similarly, properties of quantifiers bring to: if $\mathsf{Pos}(a)$ and $a \lhd U$, then $(\exists b \varepsilon U)\ \mathsf{Pos}(b)$, which we will abbreviate by $\mathsf{Pos}(U)$ from now on.

Also, the validity of $(\exists x.\phi \to \forall x.(\phi \to \psi)) \to \forall x.(\phi \to \psi)$ in intuitionistic logic shows that if $a \lhd U$ $[\mathsf{Pos}(a)]$ then $a \lhd U$. . For more details on *positivity* see [SVV96] where it is shown that it allows proofs by cases on $\mathsf{Pos}(a)$ for deductions whose conclusion is of the form $a \lhd U$.

To formulate (B_2) completely within the formal side, what we can do is to weaken $\mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{ext}(a \downarrow b)$ into

$$\frac{\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)}{\mathsf{ext}(c) \subseteq \mathsf{ext}(a \downarrow b)}$$

that is

$$\frac{c \lhd a \qquad c \lhd b}{c \lhd a \downarrow b}$$

where $c \lhd a$ and $c \lhd b$ are shorthand for $c \lhd \{a\}$ and $c \lhd \{b\}$ that we will use from now on.

But again, this is not enough: by definition $c \lhd a$ and $c \lhd b$ give $c \varepsilon a \downarrow b$ and hence $c \lhd a \downarrow b$. So, we would fail to express closure of open subsets under intersection. So, we first strengthen (B_2) to arbitrary subsets, obtaining

$$\mathsf{ext}(U) \cap \mathsf{ext}(V) \subseteq \mathsf{ext}(U \downarrow V)$$

[1] We can obtain a more general notion of constructive topological structure if we leave out the positivity predicate from this definition, and it will be not difficult to check that also taking away Pos all the results in the next sections continue to hold. On the other hand, the positivity predicate plays a main role in the definition of some particular formal topologies, like for instance the formal topology of Scott Domains (see [SVV96]).

where $U \downarrow V \equiv \cup_{a\varepsilon U, b\varepsilon V} a \downarrow b$. This holds by distributivity of $\mathcal{P}(X)$; in fact,

$$\mathsf{ext}(U) \cap \mathsf{ext}(V) \equiv \cup_{a\varepsilon U} \mathsf{ext}(a) \cap \cup_{b\varepsilon V} \mathsf{ext}(b) = \cup_{a\varepsilon U} \cup_{b\varepsilon V} \mathsf{ext}(a) \cap \mathsf{ext}(b)$$

So now by (B_2) $\mathsf{ext}(U) \cap \mathsf{ext}(V) \subseteq \cup_{a\varepsilon U} \cup_{b\varepsilon V} \mathsf{ext}(a \downarrow b)$ and hence the claim follows since ext distributes unions.

Now, the preceding idea brings to

$$\frac{\mathsf{ext}(c) \subseteq \mathsf{ext}(U) \cap \mathsf{ext}(V)}{\mathsf{ext}(c) \subseteq \mathsf{ext}(U \downarrow V)}$$

that is

$$\frac{c \lhd U \qquad c \lhd V}{c \lhd U \downarrow V}$$

which is *not* trivial. We thus arrived at the main definition.

Definition 1.3 *A* formal topology *is a triple* $\mathcal{A} \equiv (S, \lhd, \mathsf{Pos})$ *where* S *is a set,* $\lhd$ *is a relation between elements and subsets of* S*, that is*

$$a \lhd U \ \mathsf{prop}\ [a : S, U \subseteq S]$$

satisfying the following conditions:

$$\begin{array}{ll}
(\text{reflexivity}) & \dfrac{a\varepsilon U}{a \lhd U} \\[2ex]
(\text{transitivity}) & \dfrac{a \lhd U \qquad U \lhd V}{a \lhd V} \\[2ex]
(\downarrow\text{-right}) & \dfrac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}
\end{array}$$

where

$$\begin{array}{rcl}
U \lhd V & \equiv & u \lhd V \ [u\varepsilon U] \\
U \downarrow V & \equiv & \{d : S |\ (\exists u\varepsilon U)\ (d \lhd u)\ \&\ (\exists v\varepsilon V)\ (d \lhd v)\}
\end{array}$$

and Pos *is a subset of* S*, that is a propositional function over* S*, which satisfies the following conditions*

$$\begin{array}{ll}
(\text{monotonicity}) & \dfrac{\mathsf{Pos}(a) \quad a \lhd U}{(\exists b\varepsilon U)\ \mathsf{Pos}(b)} \\[2ex]
(\text{positivity}) & \dfrac{a \lhd U\ [\mathsf{Pos}(a)]}{a \lhd U}
\end{array}$$

In the following $\lhd$ *is called* cover *and* Pos positivity predicate.

In the terminology of locale theory, these structures correspond to open spaces (see [Joh84] and [Neg02], proposition 2.17).

By the preceding remarks, this axiomatic definition is satisfied by the structure induced on the formal side of any concrete topological space. However, as we explained above, its *raison d'etre* is that of gathering many more examples.

We will show that lots of examples are provided by the method of inductive definitions, starting from given axioms for the cover relation.

Actually, this method is necessary also because otherwise we would not be able, as far as we can see, to define predicatively one of the simplest constructions, namely that of co-product of formal topologies (see [NV97]). In fact, let $\mathcal{A} \equiv (S, \lhd_{\mathcal{A}}, \mathsf{Pos}_{\mathcal{A}})$ and $\mathcal{B} \equiv (T, \lhd_{\mathcal{B}}, \mathsf{Pos}_{\mathcal{B}})$ be formal topologies. We want the co-product of $\mathcal{A}$ and $\mathcal{B}$ to be a formal topology

$$\mathcal{A} + \mathcal{B} \equiv (S \times T, \lhd_{\mathcal{A}+\mathcal{B}}, Pos_{\mathcal{A}+\mathcal{B}})$$

where $S \times T$ is the usual cartesian product of sets,

$$\mathsf{Pos}_{\mathcal{A}+\mathcal{B}}((a,b)) \equiv \mathsf{Pos}_{\mathcal{A}}(a) \ \& \ \mathsf{Pos}_{\mathcal{B}}(b)$$

and $\lhd_{\mathcal{A}+\mathcal{B}}$ is the minimal cover relation satisfying the axioms

$$\begin{array}{ll} (a,b) \lhd_{\mathcal{A}+\mathcal{B}} U \times b & \text{whenever } b \in T \text{ and } a \lhd_{\mathcal{A}} U, \\ (a,b) \lhd_{\mathcal{A}+\mathcal{B}} a \times V & \text{whenever } a \in S \text{ and } b \lhd_{\mathcal{B}} V \end{array}$$

where of course $U \times b \equiv \{(u,b) : S \times T |\ u \varepsilon U\}$ and similarly for $a \times V$.

As it stands this is *not* a definition of $\lhd_{\mathcal{A}+\mathcal{B}}$ from a predicative point of view; impredicatively, $\lhd_{\mathcal{A}+\mathcal{B}}$ would simply be the intersection of all covers containing the required axioms. To solve this problem, we see no predicatively acceptable way other then that of an inductive definition.

2 Three problems and their solution

The conditions appearing in the definition of formal topology, though written in the shape of rules, must be understood as requirements of validity: if the premises hold, also the conclusion must hold. As they stand, they are by no means acceptable rules to generate inductively a cover and a positivity predicate. This is obvious if one notes that the operation $\downarrow$ among subsets, which occurs in the conclusion of $\downarrow$*-right*, is not even well defined unless we already have a complete knowledge of the cover.

The second problem is that, as we will prove in detail, admitting *transitivity* as an acceptable rule for an inductive definition is equivalent to a well-known fix-point principle, which to our knowledge does not have a predicative justification.

Thirdly, one has to make up one's mind whether the predicate Pos has to play a role in the generation of the cover or has to be added on top of it. We will see that the solution is a mixture. *Monotonicity* has an existential quantification in its conclusion, and thus we cannot expect Pos to be generated inductively. Still *positivity* plays a role in the generation of the cover.

So, to transform the axiomatic definition into good inductive rules we need to face with three problems. We discuss them in detail in this section, together with the solution we adopted, so that the reader can valuate correctly the method and the main theorem in the next section.

2.1 Formal topologies with pre-order

As we reminded above, the definition of the operation $\downarrow$ among subsets depends on the covers and it requires the cover to be known. However, a crucial observation is that only the trace of the cover on elements is sufficient. The idea is then to separate covers between elements, that is $a \lhd b$, from those $a \lhd U$ with an arbitrary subset U on the right, so that we can block the former, require $\downarrow$ on it and then generate the latter. So, we must add, to those of a formal topology, an extra primitive expressing what in the concrete case is $\mathsf{ext}(a) \subseteq \mathsf{ext}(b)$. We can obtain this in two technically different ways. The easiest way is then to add directly a pre-order relation $a \leq b$ among observables. The other is to add a binary operation $\bullet$ between observables, called combination, whose interpretation is that $\mathsf{ext}(a \bullet b) = \mathsf{ext}(a) \cap \mathsf{ext}(b)$, so that $\mathsf{ext}(a) \subseteq \mathsf{ext}(b)$ corresponds to $a \bullet b = a$. In this paper we will consider only the first solution, even if an analogous development is possible for the second one.

$\lhd$.

Adopting a pre-order $\leq$ as a new primitive, the natural definition is:

Definition 2.1 *A* formal topology with pre-order*, shortly a* $\leq$-formal topology*, is a quadruple* $\mathcal{A} \equiv (S, \leq, \lhd, \mathsf{Pos})$ *where* S *is a set,* $\leq$ *is a pre-order relation over* S*, that is* $\leq$ *is reflexive and transitive,* $\lhd$ *is a relation between elements and subsets of* S *which satisfies the following conditions*

$$\text{(reflexivity)} \quad \frac{a \varepsilon U}{a \lhd U}$$

$$\text{(transitivity)} \quad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

$$(\leq\text{-left}) \quad \frac{a \leq b \qquad b \lhd U}{a \lhd U}$$

$$(\leq\text{-right}) \quad \frac{a \lhd U \qquad a \lhd V}{a \lhd \downarrow U \cap \downarrow V}$$

where $\downarrow U \equiv \{c : S | \ (\exists u \varepsilon U) \ c \leq u\}$ *and* Pos *is a positivity predicate with respect to* $\lhd$*.*

The condition $\leq$-*left* is clearly equivalent to the fact that $\leq$ respects $\lhd$, that is

$$\frac{a \leq b}{a \lhd b}$$

Of course, we must not require $\leq$ to coincide with $\lhd$ on elements, otherwise this would bring us back to the problem of the definition of $U \downarrow V$.

Since $\leq$ respects $\lhd$, for any subset U we have $\downarrow U \subseteq \downarrow^{\lhd} U$, where $\downarrow^{\lhd} U \equiv \{c : S | \ (\exists u \varepsilon U) \ c \lhd u\}$. Thus $\downarrow U \cap \downarrow V \subseteq \downarrow^{\lhd} U \cap \downarrow^{\lhd} V \equiv U \downarrow V$, so that $\leq$-*right* implies $\downarrow$-*right*. Thus any $\leq$-formal topology is a formal topology. The converse is trivial: given any formal topology $(S, \lhd, \mathsf{Pos})$, all we need to do is to define

$$a \leq b \equiv a \lhd b$$

and we obviously obtain a $\leq$-formal topology with a cover and a positivity predicate coinciding with the original ones.

Since we will deal almost exclusively with the operation $\downarrow U \cap \downarrow V$ rather than $\downarrow^{\lhd} U \cap \downarrow^{\lhd} V$, in any $\leq$-formal topology we will abbreviate $\downarrow U \cap \downarrow V$ with $U \downarrow V$. There is little danger of confusion with the previous definition of $U \downarrow V$, since in that case we can understand it as defined through the pre-order $a \leq b \equiv a \lhd b$.

Other equivalent formulations are possible of the previous definition 2.1. Here we mention just one, to be used in the following.

Lemma 2.2 *For any cover relation $\lhd$ closed under* reflexivity*,* transitivity *and* $\leq$-left *the condition* $\leq$-right *is equivalent to the following*

$$\textit{(localization)} \qquad \frac{a \lhd U}{a \downarrow b \lhd U \downarrow b}$$

where $a \downarrow b \equiv \{c \in S \mid (c \leq a) \ \& \ (c \leq b)\}$.

Proof. Immediate.

2.2 The problem of *transitivity*

An inductive definition of a cover will start from some axioms, which at the moment we assume to be given by means of any infinitary relation

$$R(a, U) \ \mathsf{prop} \ [a : S, U \subseteq S]$$

We thus want to generate the least cover $\lhd_R$ which satisfies

$$\text{(axioms)} \qquad \frac{R(a, U)}{a \lhd_R U}$$

As we will see, the task of forcing $\lhd_R$ to satisfy $\leq$*-left* and $\leq$*-right* is essentially only technical and not too difficult, once it is clear that $\lhd_R$ satisfies *reflexivity* and *transitivity*. So we concentrate in this section on the conceptual problem of constructing the minimal infinitary relation $\lhd_R$ which satisfies *reflexivity*, *transitivity* and the axioms given by R.

¿From an impredicative point of view, $\lhd_R$ is easily obtained "from above" simply as the intersection of the collection $\mathcal{C}_R$ of all the reflexive, transitive infinitary relations containing R. In fact, it is clear that the total relation is in $\mathcal{C}_R$ and that the intersection preserves all such conditions. Even impredicatively, however, this is not enough to say that $\lhd_R$ is defined inductively; to be able to prove a property P by induction on the generation of $\lhd_R$, that is by showing that P contains R and is preserved by *reflexivity* and *transitivity*, one still needs a justification. In fact, one cannot a priori exclude that there is some rule which is valid in all the infinitary relations in $\mathcal{C}_R$, but which is not derivable from the axioms by means of *reflexivity* and *transitivity*.

¿From a classical point of view, one can easily prove that this is not the case. In fact, by using the axiom of choice one can construct a list of all of the subsets of S and then one can "correct" it in such a way that any subset appears in the list an infinite number of times, that is after any occurrence of a subset in the list there is still another later. Let us denote this list by $V_1, V_2, \ldots, V_\omega, \ldots$. Now consider the following inductive definition of $\lhd_R$:

$$\begin{array}{lll} \lhd_0 & = & R \cup \{(a,U) : a \varepsilon U\} \\ \lhd_{\alpha+1} & = & \lhd_\alpha \cup \{(a,U) : a \lhd_\alpha V_\alpha \ \& \ V_\alpha \lhd_\alpha U\} \\ \lhd_\beta & = & \cup_{\alpha<\beta} \lhd_\alpha \end{array}$$

and hence

$$\lhd_R = \cup_{\alpha<\lambda} \lhd_\alpha$$

where λ is the ordinal of the set of $\mathcal{P}(S)$: what one has to do is to check an infinite number of times all of the subsets of S. It is clear that the relation $\lhd_R$ inductively defined in such a way satisfies all the conditions and nothing more.

Note however that ordinals are not really necessary to prove the existence of the minimal cover relation since, as we will see, it is possible to obtain the same result also for an impredicative set theory by using only intuizionistic logic.

Predicatively the method of defining $\lhd_R$ as the intersection $\mathcal{C}_R$ is not acceptable, since there is no way of producing $\mathcal{C}_R$ above as a set-indexed family and hence to define its intersection.

Therefore, we must obtain $\lhd_R$ "from below" by means of some introductory rules. The first naive idea is that of using axioms, *reflexivity* and *transitivity* for this purpose. But then a serious problem emerges: in the premises of *transitivity*, that is

$$\frac{a \lhd_R V \qquad V \lhd_R U}{a \lhd_R U}$$

there is a subset V which does not appear in the conclusion. This means that the tree of possible premises to conclude that $a \lhd_R U$ has an unbounded branching: each subset V satisfying $a \lhd_R V$ and $V \lhd_R U$ would be enough to obtain $a \lhd_R U$, and there is no way to survey them all. Also, a dangerous vicious circle seems to be present: the subset V, whose existence would be enough to obtain $a \lhd_R U$, could be defined by means of the relation $\lhd_R$ itself which we are trying to construct. In this way, the instructions to try to build up $\lhd_R$ would not be fixed in advance, but change along their application.

Some of us have tried for some time to eliminate *transitivity*, by reducing it to other less problematic rules. We convinced ourselves, however, that this is an unrealistic expectation. In fact, $\lhd$ can be read as a logical consequence relation on the axioms given by R, and then *transitivity* plays the role of *cut*, where what is cut is the subset V. So, a general method to eliminate *transitivity* for any relation R would correspond to a general theorem of cut-elimination for all theories, and we know that this is impossible. In fact, we know how sensitive cut-elimination is to the way the theory, that is R, is presented; in this sense, the solution we will give in section 3 can be seen as a cut-elimination theorem of remarkable generality.

All these, though convincing, are not yet conclusive arguments. For this reason, following a suggestion by P. Aczel, we now show in detail that the problem of *transitivity* is reducible to that of the existence of the least fix-point for monotone operators, which is better known and seems to resist to a predicative justification.

We begin by reducing the problem of *transitivity* to its essence. The connection with proof theory, in the form of the analogy between *transitivity* and cut, suggests a considerable reduction. Thinking of an application of *transitivity* as an application of cut, and hence the construction of $\lhd_R$ as a derivation in a proof-system (with our axioms, *reflexivity* and *transitivity* as the only inference rules), one can see that *transitivity* can be lifted until all its applications are only of the special form

$$\dfrac{\dfrac{R(a,V)}{a \lhd_R V}\,axiom \qquad V \lhd_R U}{a \lhd_R U}\,trans.$$

which corresponds to cut on axioms. In fact, given a figure like

$$\dfrac{\dfrac{a \lhd_R W \quad W \lhd_R V}{a \lhd_R V}\,trans. \qquad V \lhd_R U}{a \lhd_R U}\,trans.$$

one can reduce it to

$$\dfrac{a \lhd_R W \qquad \dfrac{W \lhd_R V \quad V \lhd_R U}{W \lhd_R U}\,trans.}{a \lhd_R U}\,trans.$$

where one application of *transitivity* has been moved from the left branch to the right branch. In this way, the number of application of *transitivity* in the left branch is lowered, and by iterating the reduction in the left branch we either reach a figure of the form

$$\dfrac{\dfrac{a\varepsilon W}{a \lhd_R W} \quad W \lhd_R V}{a \lhd_R V}$$

which, by definition of $W \lhd_R V$, is immediately reduced to $a \lhd_R V$ with no application of *transitivity*, or an application of *transitivity on axioms.* Like with cut-elimination, by iterating such reduction on all applications of *transitivity*, we reach a proof were *transitivity* is applied only to the axioms.

Therefore one could think that a good strategy to get rid of *transitivity* would be to adopt only

$$\text{(reflexivity)} \qquad \frac{a\varepsilon U}{a \lhd_R U}$$

$$\text{(trax)} \qquad \frac{R(a,V) \qquad V \lhd_R U}{a \lhd_R U}$$

as introduction rules to generate $\lhd_R$. In fact, axioms would be derivable because obviously $V \lhd_R V$ holds by *reflexivity* and hence $a \lhd_R V$ follows from $R(a, V)$ by *trax*, and by the above argument *transitivity* would be admissible in this formal system.

To complete this argument into a proof, we should argue by induction on the generation of $\lhd_R$. However, one can see immediately that even adopting *trax*, rather then *transitivity*, does not change the conceptual essence of the problem of justifying induction through the generation of $\lhd_R$. The implicit quantification on subsets, and thus the unbounded branching and the vicious circle, have remained, since we have passed simply from

$$(\exists V \subseteq S)\ ((a \lhd_R V \ \&\ V \lhd_R U)) \rightarrow a \lhd_R U$$

to

$$(\exists V \subseteq S)\ ((R(a, V) \ \&\ V \lhd_R U)) \rightarrow a \lhd_R U$$

The idea of reducing to *trax*, however, allows to see more easily how the principle of the least fix-point for monotone operators comes in.

2.2.1 Fix-points and saturation

An *operator* on subsets is any function $F : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$, that is a map bringing subsets of S into subsets of S and respecting extensional equality of subsets. F is called *monotone* if $U \subseteq V \rightarrow F(U) \subseteq F(V)$.

First of all we need to recall the correspondence between infinitary relations and operators on subsets. Given the infinitary relation

$$R(a, U)\ \mathsf{prop}\ [a : S, U \subseteq S]$$

we define the operator $F_R : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ by putting

$$F_R(U) \equiv \{a \in S \mid R(a, U)\}$$

Conversely, given an operator $F : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$, we define the relation R_F by putting

$$R_F(a, U) \equiv a \varepsilon F(U)$$

Note that the correspondence is clearly biunivocal; actually, the move from R to F_R is simply abstraction on the variable a, and conversely the move from F to R_F is just application to a. So, infinitary relations and operators on subsets are just two different notations for one and the same mathematical content, and we call "rewriting" to pass from one to the other. Thus, if R is associated with F, we say that $R(a, U)$ is a rewriting of $a \varepsilon F(U)$. Note that rewriting $U \subseteq F(V)$ one obtains $(\forall a \varepsilon U)\ R(a, V)$; so, when F is associated with $\lhd$, $U \subseteq F(V)$ is a rewriting of $U \lhd V$.

By rewriting, we immediately see that an operator F is monotone if and only if, for the corresponding relation R, $R(a, V)$ and $V \subseteq U$ yield $R(a, U)$; thus we say that in this case R is monotone.

Again by rewriting, we easily see that an infinitary relation $\lhd$ satisfies *reflexivity* and *transitivity* if and only if the corresponding operator F is a closure operator, that is $U \subseteq F(U)$, $U \subseteq V \to F(U) \subseteq F(V)$ and $F(F(U)) \subseteq F(U)$ for any $U, V \subseteq S$. In fact, rewriting *reflexivity* gives $U \subseteq F(U)$ and rewriting *transitivity* gives $U \subseteq F(V) \to F(U) \subseteq F(V)$; together they are equivalent to F being a closure operator.

The connection with least fix-points of monotone operators is now easily seen. First, let us recall that a subset Z is called the *least fix-point* for an operator F if Z is a fix-point for F, that is $F(Z) = Z$, and any other fix-point contains Z, that is $F(W) = W$ yields $Z \subseteq W$.

F.

Theorem 2.3 *Assume that for any infinitary relation R, a relation $\lhd_R$ can be obtained inductively by closing R under* reflexivity *and* trax*, that is, assume that a relation $\lhd_R$ exists which, for any subset U and any property P, satisfies:*

$$\text{(a)} \quad \frac{a \varepsilon U}{a \lhd_R U}$$

$$\text{(b)} \quad \frac{R(a, V) \quad V \lhd_R U}{a \lhd_R U}$$

$$\text{(c)} \quad \frac{a \lhd_R U \quad U \subseteq P \quad x \varepsilon P \ [R(x, V), V \subseteq P]}{a \varepsilon P}$$

Then, for any monotone operator F, the least fix-point of F exists.

Proof. Given any monotone operator F, let R be the corresponding relation, that is $R(a, V) \equiv a \varepsilon F(V)$, and apply the assumption to such R to obtain $\lhd_R$. Then $Z \equiv \{a \in S|\ a \lhd_R \emptyset\}$ is the least fix-point of F. In fact

(1) $F(Z) \subseteq Z$

holds, because $a \varepsilon F(Z) \equiv R(a, Z)$ and $Z \lhd_R \emptyset$ holds by definition, so that, by (b), also $a \lhd_R \emptyset$, that is $a \varepsilon Z$. Moreover

(2) $F(W) \subseteq W \to Z \subseteq W$

is easily proved by (c). In fact, assume $F(W) \subseteq W$ and let $a \varepsilon Z$, that is $a \lhd_R \emptyset$. Trivially $\emptyset \subseteq W$, so to obtain $a \varepsilon W$ by (c) it is enough to show that, for any $x \in S$, $x \varepsilon W$ follows from $R(x, V)$ and $V \subseteq W$. Since F is monotone, $V \subseteq W$ gives $F(V) \subseteq F(W)$ and hence $F(V) \subseteq W$ because $F(W) \subseteq W$; so from $R(x, V) \equiv x \varepsilon F(V)$ it follows that $x \varepsilon W$.

The proof is now quickly concluded. From (1) it follows that $F(F(Z)) \subseteq F(Z)$ by monotonicity of F, hence by (2) also $Z \subseteq F(Z)$, which with (1) gives $Z = F(Z)$. And (2) gives $F(W) = W \to Z \subseteq W$ a fortiori.

It is a fact that a predicative justification of the existence of least fix-points for monotone operators has not been given yet, and some scholars believe that actually it cannot be given. We agree with them. So, by the above proposition, the same predicament applies to the expectation that $\lhd_R$ can be defined inductively by closing under *reflexivity* and *trax* (or *transitivity*). The way out we propose will be treated in section 3.

Here we continue our analysis of the relation between existence of least fixpoints for monotone operators and inductive generation via *transitivity*. We will justify, at least impredicatively, inductive generation via *transitivity* and we will improve the understanding of theorem 2.3 above.

Given any infinitary relation R, we say that a subset U is *R-saturated* if it satisfies

$$\frac{R(a,V) \qquad V \subseteq U}{a \varepsilon U}$$

We say that Z is the *R-saturation* of U if Z is the least R-saturated subset containing U, that is, Z is R-saturated, $U \subseteq Z$ and whenever $U \subseteq W$, for some R-saturated subset W, then $Z \subseteq W$.

Now, for any given relation R, assume that $\lhd_R$ exists which satisfies (a), (b) and (c) in theorem 2.3 and let $\mathcal{R}$ be the operator associated with $\lhd_R$; by rewriting $a \lhd_R U$ as $a \varepsilon \mathcal{R}(U)$, the conditions (a), (b) and (c) are immediately seen to be equivalent to

$$(\mathrm{a}') \quad U \subseteq \mathcal{R}(U)$$

$$(\mathrm{b}') \quad \frac{R(a,V) \quad V \subseteq \mathcal{R}(U)}{a \varepsilon \mathcal{R}(U)}$$

$$(\mathrm{c}') \quad \frac{U \subseteq P \quad x \varepsilon P\ [R(x,V), V \subseteq P]}{\mathcal{R}(U) \subseteq P}$$

Clearly (b$'$) says that $\mathcal{R}(U)$ is R-saturated, (a$'$) that it contains U, and (c$'$) that it is the least such. Thus considering *reflexivity* and *trax* as good inductive rules is equivalent to the

Principle of least R-saturation
For any infinitary relation $R(a,V)$ prop $[a : S, V \subseteq S]$ and any subset U, there exists a subset $\mathcal{R}(U)$ which is the least R-saturation of U.

Now, let R be any infinitary relation. Then by "monotonization" of R we mean the minimal infinitary relation R^* which is monotone and contains R. From an impredicative point of view R^* is defined by

$$R^*(a,U) \equiv (\exists V \subseteq S)\ (R(a,V)\ \&\ V \subseteq U)$$

In fact, R^* is obviously monotone and contains R. Moreover, it is clearly the least monotone relation containing R. Of course, if R is a monotone relation no impredicative definition is required to define R^* and in this case all the results to follow will have a predicative proof. It is interesting to note that the cover relation $\lhd$ generated by R and R^* is exactly the same; in fact $\lhd$ is the minimal infinitary relation obtained by closing R under *reflexivity* and *transitivity* and hence it is also a monotone relation; thus if $\lhd$ contains R then it also contains R^* and the result follows by minimality. V)

Given any infinitary relation R let us now consider the operator Φ associated to its monotonization, that is

$$a \varepsilon \Phi(U) \text{ if and only if } (\exists V \subseteq S)\ (R(a,V)\ \&\ V \subseteq U)$$

Note that Φ is just the operator associated with R if R is monotone. Then (a′), (b′) and (c′) above become:

(a″) $U \subseteq \mathcal{R}(U)$

(b″) $\Phi(\mathcal{R}(U)) \subseteq \mathcal{R}(U)$

(c″) If $U \subseteq P$ and $\Phi(P) \subseteq P$ then $\mathcal{R}(U) \subseteq P$

For any operator F, we say that Z is the *least pre-fix-point containing* U if $U \subseteq Z$, $F(Z) \subseteq Z$ and $(U \subseteq W\ \&\ F(W) \subseteq W) \to Z \subseteq W$. So (a″), (b″) and (c″) say that $\mathcal{R}(U)$ is the least pre-fix-point of Φ containing U. So the equivalence between (a′), (b′) and (c′) and (a″), (b″) and (c″) says that for any infinitary relation R and any subset U, the least R-saturation of U coincides with the least pre-fix-point of Φ containing U.

Note that the least pre-fix-point of F containing U coincides with the least pre-fix-point (containing $\emptyset$) of the operator $F^U(W) \equiv U \cup F(W)$, and that F^U is monotone if so is F. And finally, it is easy to check that, for a monotone operator, the least pre-fix-point is actually the least fix-point (see the last lines of the proof of theorem 2.3). Summing up, we have given a proof of the following theorem, whose only (possible) impredicative step is a monotonization of an infinitary relation.

Theorem 2.4 *The principle of least R-saturation for any infinitary relation R is equivalent to the existence of the least fix-point for any monotone operator F.*

It is known that existence of least fix-points can be proved also in a non-classical foundation, like topos theory. Topos theory is often considered the foundation for the development of locale theory, or pointless topology (see [Joh82], [JT84]). So, the meaning of the above theorem is that it makes genuinely inductive methods explicitly available in pointless topology.

2.3 Dealing with the positivity predicate

The definition of formal topology includes, besides a cover $\lhd$, a positivity predicate Pos. The two conditions that we require on Pos are different in nature. *Monotonicity* is a condition of closure of Pos with respect to the cover, but has nothing to do with its generation since its conclusion is a proposition on the positivity predicate, and in this sense it is a "static" condition on Pos. On the other hand, *positivity* is a condition also on the cover and its conclusion is about the cover relation, and thus it contributes to the generation of the cover.

To obtain *monotonicity*, we will find out some conditions which a given predicate Pos must satisfy *before* the cover is generated, so that it becomes monotonic with respect to the cover, *after* it is generated. To get an idea, assume that an infinitary relation R and a predicate Pos are given which satisfy

$$\text{(monotonicity on axioms)} \quad \frac{\mathsf{Pos}(a) \qquad R(a,V)}{(\exists b \varepsilon V)\ \mathsf{Pos}(b)}$$

If we could generate the cover by *reflexivity* and *trax*, we would easily prove *monotonicity* by induction. In fact, if $\mathsf{Pos}(a)$ and $a \varepsilon U$, then trivially $\mathsf{Pos}(U)$. And if $\mathsf{Pos}(a)$, $R(a,V)$ and $V \lhd U$ then by *monotonicity on axioms* there exists $b \varepsilon V$ such that $\mathsf{Pos}(b)$ and, whatever b is, $\mathsf{Pos}(U)$ follows from $V \lhd U$ by inductive hypothesis.

After the results of the previous section 2.2, we know that we must use other rules to generate covers; the idea to obtain monotonicity will however remain the same, though some technical complications will be necessary.

Then, there are conditions which depends on the particular presentation that we are going to use for formal topologies. For instance if we want to deal with $\leq$-formal topologies it is clear that the following condition must hold, because $a \leq b$ yields $a \lhd b$:

$$\text{(monotonicity on } \leq) \quad \frac{\mathsf{Pos}(a) \qquad a \leq b}{\mathsf{Pos}(b)}$$

To impose *positivity*, we will simply put it among the rules generating the cover, that is we simply add the following rule

$$\text{(positivity rule)} \quad \frac{a \lhd U\ [\mathsf{Pos}(a)]}{a \lhd U}$$

We will see in section 3.2 that, as far as predicativity is concerned, it is as safe as the other rules that we will adopt.

3 Inductive generation

The problem concerning *transitivity*, also in its reduced form of transitivity on axioms, is essentially due to the fact that it allows to infer $a \lhd U$ from $R(a,V)$ and $V \lhd U$, whatever the subset V is. Thus the possible premise of $a \lhd U$ cannot be indexed by a set: the validity of $a \lhd U$ depends on an existential quantification on $\mathcal{P}(S)$, namely $(\exists V \subseteq S)\ (R(a,V)\ \&\ V \lhd U)$. The solution is simply to reduce it to a quantification over a set, so that the branching is under control. The most general case we are able to devise is then to have a family of sets $I(a)$ set $[a : S]$, so that the previous quantification over $\mathcal{P}(S)$ to infer $a \lhd U$ will become a quantification over $I(a)$, and for each $i \in I(a)$ a subset $C(a,i) \subset S$, which will play the role previously played by the subset V. So, the subsets which are postulated to cover a given element a are not given as those V for which $R(a,V)$ holds, but directly as the family $C(a,i)$ indexed on the set $I(a)$. In this way the dependence on the general notion of subset is avoided, the axioms are surely not affected by the process of generation and any danger of vicious circles is stopped.

We will see in section 4 that the restriction is not too severe, since it is met by most of the known examples of formal topologies. Actually, proving that a specific formal topology is not included is not simple: we do this in the end of section 4.

3.1 Set-based axioms and set-based relations

Let S be a set. We say that a set indexed family $I(a)$ set $[a : S]$ together with a family of subsets $C(a,i) \subseteq S$ $[a : S, i : I(a)]$ is an *axiom-set*. The intended meaning is that, for all $a \in S$, the subset $C(a,i)$ is assumed to be a cover of a, for any $i \in I(a)$. We can think of such axioms also as an infinitary relation R, linking a with $C(a,i)$ for any $i \in I(a)$, that is, the relation $R(a,V)$ holds if and only if there exists $i \in I(a)$ such that $V =^S C(a,i)$.

An application of the rule *trax* for such relation is particularly simple, since the assumption that a is related with any $C(a,i)$ can be understood; so we reach the rule

$$\text{(infinity)} \qquad \frac{i \in I(a) \qquad C(a,i) \lhd U}{a \lhd U}$$

Note that the previous implicit quantification over $\mathcal{P}(S)$ has now become a quantification over the family $C(a,i) \subseteq S$ $[i \in I(a)]$, which is reduced to a quantification over $I(a)$. There remains the problem that the right premise $C(a,i) \lhd U$ of *infinity* contains a subset at the left. We now must begin to be more careful in the analysis of derivations, and thus we understand that $C(a,i) \lhd U$ is an abbreviation for a derivation of $x \lhd U$ from $x \varepsilon C(a,i)$ with the variable x free. So the expanded formulation of infinity is

$$\text{(infinity)} \qquad \frac{i \in I(a) \qquad x \lhd U\ [x \varepsilon C(a,i)]}{a \lhd U}$$

and we use the previous formulation as an abbreviation of this. We understand that a similar convention applies to all rules to follow, which contain a subset at the left of $\lhd$.

Let us go back to the relation linked with an axiom-set. A bit more generally, we add monotonicity and say that an infinitary relation R is *set-based* if there exist two families I and C as above such that, for all $a \in S$ and $V \subseteq S$,

$$R(a,V) \text{ if and only if } (\exists i \in I(a))\ C(a,i) \subseteq V$$

We can now see immediately that the problem of closing under *transitivity* is solved for set-based relations. In fact, given any two families

$$I(a)\ \mathsf{set}\ [a : S]$$

$$C(a,i) \subseteq S\ [a : S, i : I(a)]$$

we define $\lhd$ inductively by the rules

$$\text{(reflexivity)} \qquad \frac{a \varepsilon U}{a \lhd U}$$

$$\text{(infinity)} \qquad \frac{i \in I(a) \qquad C(a,i) \lhd U}{a \lhd U}$$

Such rules fall under a general scheme for which a predicative justification has already been given (see [Dyb94]); they are for instance an example of the Tree type in [NPS90]. So we know that also the elimination rule

$$\frac{a \lhd U \qquad U \subseteq P \qquad x\varepsilon P\ [i : I(x), C(x,i) \subseteq P]}{a\varepsilon P}$$

is valid. This means that proofs by induction on *reflexivity* and *infinity* are justified.

It is now easy to prove by induction that:

Theorem 3.1 *For any infinitary relation R which is set-based on I and C as above, the relation $\lhd$ defined inductively by* reflexivity *and* infinity *is the least infinitary relation which contains R and is closed under* reflexivity *and* transitivity.

Proof. First, we show that the rules generating $\lhd$ are valid, in the sense that they hold for any relation $\lhd'$ which contains R and is closed under *reflexivity* and *transitivity*. Trivially *reflexivity* is valid. And if $C(a,i) \lhd' U$ for some $i \in I(a)$, then also $a \lhd' U$ by *transitivity* on the axiom $R(a, C(a,i))$; this shows that *infinity* is valid.

Then we show that they are complete, in the sense that they allow to derive $a \lhd U$ whenever it holds for each reflexive, transitive $\lhd'$ containing R. That is, we prove by induction that $\lhd$ is indeed closed under *reflexivity* and *transitivity* and that it contains R. Closure under *reflexivity* is built in the definition. Closure under *transitivity* is proved by induction on the derivation of the left premise $a \lhd U$, the right premise being $U \lhd V$. If $a \lhd U$ is obtained by *reflexivity* from $a\varepsilon U$ then $a \lhd V$ follows from $U \lhd V$, since by definition $U \lhd V$ means $x \lhd V\ [x\varepsilon U]$. If $a \lhd U$ is obtained by *infinity* from $C(a,i) \lhd U$ for some $i \in I(a)$, then by inductive hypothesis $C(a,i) \lhd V$, from which $a \lhd V$ by *infinity*.

Finally, $R(a,U)$ by assumption means that there exists $i \in I(a)$ such that $C(a,i) \subseteq U$; then by *reflexivity* $C(a,i) \lhd U$ and hence $a \lhd U$ by *infinity*.

It is interesting to note that, as a corollary of a general theorem on deductive systems by P. Aczel [Aczel], we have also the following result.

Theorem 3.2 *If R is set-based, then also the least infinitary relation closed under reflexivity and transitivity containing R is set-based.*

We will recall the proof of this theorem and we will use it in section 4.7.

3.2 Inductive generation of formal topologies

In order to generate a cover inductively it is enough to modify the generation process of the previous section by forcing the resulting relation $\lhd$ to satisfy the condition $\downarrow$-right. As shown in section 2.1, to this aim one must restrict to formal topologies with an extra primitive. We deal here with formal topologies with a preorder $\leq$ and thus we have to force $\leq$-*left* and $\leq$-*right* to hold. By lemma 2.2 we can equivalently force $\leq$-*left* and *localization*. Then the idea for the solution comes from the following remark:

Localization Lifting
Every application of *localization* can be "lifted over" any application of *reflexivity*, *transitivity* and $\leq$-*left*.

The suitable proof transformations are shown by the following figures:

$$\dfrac{\dfrac{a\varepsilon U}{a \lhd U}\ \textit{refl.}}{a \downarrow c \lhd U \downarrow c}\ \textit{loc.} \quad \rightsquigarrow \quad \dfrac{\dfrac{a\varepsilon U}{a \downarrow c \subseteq U \downarrow c}}{a \downarrow c \lhd U \downarrow c}\ \textit{refl.}$$

$$\dfrac{\dfrac{a \lhd V \quad V \lhd U}{a \lhd U}\ \textit{trans.}}{a \downarrow c \lhd U \downarrow c}\ \textit{loc.} \quad \rightsquigarrow \quad \dfrac{\dfrac{a \lhd V}{a \downarrow c \lhd V \downarrow c}\ \textit{loc.} \quad \dfrac{V \lhd U}{V \downarrow c \lhd U \downarrow c}\ \textit{loc.}}{a \downarrow c \lhd U \downarrow c}\ \textit{trans.}$$

$$\dfrac{\dfrac{a \leq b \quad b \lhd U}{a \lhd U}\ \leq\textit{-left}}{a \downarrow c \lhd U \downarrow c}\ \textit{loc.} \quad \rightsquigarrow \quad \dfrac{\dfrac{a \leq b}{a \downarrow c \subseteq b \downarrow c} \quad \dfrac{b \lhd U}{b \downarrow c \lhd U \downarrow c}\ \textit{loc.}}{a \downarrow c \lhd U \downarrow c}$$

So, in a proof figure which contains only *reflexivity*, *transitivity* and $\leq$-left, *localization* can be lifted until it is applied only under the axioms; thus to obtain closure under *localization* we could either restrict its application or simply require that the axioms are closed under localization. Because of the problem with *transitivity* we cannot use this approach in an inductive process of generation, but it suggests how to modify the inductive generation with *reflexivity* and *infinity* to obtain a relation $\lhd$ which is closed under *localization*. Applying the idea of *loc-lifting* to a figure of the form

$$\dfrac{\dfrac{i \in I(a) \quad C(a,i) \lhd V}{a \lhd V}\ \textit{infinity}}{a \downarrow c \lhd V \downarrow c}\ \textit{loc.}$$

we would apply *localization* to $C(a,i) \lhd V$ obtaining $C(a,i) \downarrow c \lhd V \downarrow c$, but then we could not apply *infinity* in its present form. However, we can modify *infinity* into a valid rule which includes *localization* on the left. We obtain

$$\dfrac{i \in I(a) \qquad C(a,i) \downarrow c \lhd V \downarrow c}{a \downarrow c \lhd V \downarrow c}$$

and thus we reach

$$\text{(loc-infinity)} \qquad \dfrac{i \in I(a) \qquad C(a,i) \downarrow c \lhd U}{a \downarrow c \lhd U}$$

which is still valid in any cover satisfying $a \lhd C(a,i)$. In fact, from $a \lhd C(a,i)$ one has $a \downarrow c \lhd C(a,i) \downarrow c$ by *localization*, and hence the premise $C(a,i) \downarrow c \lhd U$

gives $a \downarrow c \lhd U$ by *transitivity*. It is easy to check that this rule permutes with *localization*. However, it has the drawback that a subset appears on the left of its conclusion, and this could cause complications in a rigorous formalization of proofs by induction. We can then write *loc-infinity* in the equivalent form

$$\frac{x \leq c \qquad x \leq a \qquad i \in I(a) \qquad C(a,i) \downarrow c \lhd U}{x \lhd U}$$

which, when x is taken to be c itself, gives

$$\frac{c \leq a \qquad i \in I(a) \qquad C(a,i) \downarrow c \lhd U}{c \lhd U}$$

We can see that actually this special case is enough to give back the full rule of *loc-infinity*. In fact, assume $x \leq c$, $x \leq a$ and $C(a,i) \downarrow c \lhd U$. Since $x \leq c$ implies $\downarrow x \subseteq \downarrow c$ and hence $C(a,i) \downarrow x \subseteq C(a,i) \downarrow c$, we obtain $C(a,i) \downarrow x \lhd U$; together with $x \leq a$, this allows to obtain $x \lhd U$ by the special case.

We thus choose the special case, since it has one premise less then *loc-infinity*, and write it down as usual with $a \lhd U$ as a conclusion:

$$(\leq\text{-infinity}) \qquad \frac{a \leq b \qquad i \in I(b) \qquad C(b,i) \downarrow a \lhd U}{a \lhd U}$$

We will show that $\leq$-*infinity*, together with *reflexivity* and $\leq$-*left*, is sufficient to generate the least cover satisfying the axioms. But if we wish to generate a formal topology, according to the present definition, we must have also a positivity predicate Pos. We remarked in section 2.3 that we must start from a given predicate $\mathsf{Pos}(a)$ prop $[a : S]$ which is assumed to be monotonic with respect to the axioms. But now the axioms are fitted in $\leq$-*infinity*, and hence we are brought to require

$$(\text{monotonicity on } \leq\text{-infinity}) \qquad \frac{\mathsf{Pos}(a) \qquad a \leq b \qquad i \in I(b)}{\mathsf{Pos}(a \downarrow C(b,i))}$$

It is immediate to see that this condition has to hold if we want the positivity predicate to be monotone. In fact from $a \leq b$ we can deduce $a \lhd a \downarrow C(b,i)$ for any $i \in I(b)$, by using $\leq$-*left* and $\leq$-*right*, and hence if $\mathsf{Pos}(a)$ it must also be $\mathsf{Pos}(a \downarrow C(b,i))$. We will prove that *monotonicity* of Pos with respect to the generated cover follows.

On the other hand, the condition of *positivity* is put in the process of generation itself, that is we add

$$(\text{positivity}) \qquad \frac{a \lhd U \ [\mathsf{Pos}(a)]}{a \lhd U}$$

to the rules generating $\lhd$. The premise of *positivity* is equivalent to

$$x \lhd U \ [\mathsf{Pos}(x) \ \& \ x = a]$$

and hence, after putting

$$a^+ \equiv \{x \in S | \ \mathsf{Pos}(x) \ \& \ x = a\}$$

also to $a^+ \lhd U$. So *positivity* can be formulated as

$$\frac{a^+ \lhd U}{a \lhd U}$$

which shows that it falls under the same schema that we have chosen above for axioms. More precisely, if I and C are an axiom-set, we can define I' by adding a new element $\sharp$ to each $I(a)$ and C' by putting $C'(a,\sharp) \equiv a^+$ and $C'(a,i) \equiv C(a,i)$ for any $i \in I(a)$. Then the cover generated by I' and C' will be the least cover which satisfies *positivity* and which contains the cover generated by I and C.

We are finally ready to state and prove the main theorem:

Theorem 3.3 (Inductive generation of formal topologies) *Let S be any set, $\leq$ any pre-order on S and let $I(a)$* set $[a : S]$ *and $C(a,i) \subseteq S$ $[a : S, i : I(a)]$ be an axiom-set.*

Then the infinitary relation $\lhd_0$ defined inductively by the rules reflexivity, $\leq$-left *and* $\leq$-infinity *is the least cover satisfying $a \lhd_0 C(a,i)$ $[a : S, i : I(a)]$.*

Assume in addition that a predicate $\mathsf{Pos}(a)$ set $[a : S]$ *is given which satisfies* monotonicity on $\leq$-infinity *and* monotonicity on $\leq$ *and let $\lhd$ be the infinitary relation generated by the three rules above plus* positivity.

Then $\mathcal{A} \equiv (S, \leq, \lhd, \mathsf{Pos})$ is a $\leq$-formal topology and $\lhd$ is the least among the covers $\lhd'$ satisfying $a \lhd' C(a,i)$ $[a : S, i : I(a)]$ and making $(S, \leq, \lhd', \mathsf{Pos})$ a formal topology.

Proof. We already showed that all the rules that we use in the generation process are valid, thus we have only to show that they are complete, in the sense that they allow to derive $a \lhd_0 U$ (and $a \lhd U$) whenever it holds for a cover (formal topology) satisfying the axioms. This amounts to prove that $\lhd_0$ ($\lhd$) is closed under *reflexivity*, *transitivity*, *$\leq$-left* and *$\leq$-right* (and *positivity*) and that, for any $a \in S$ and $i \in I(a)$, $a \lhd C(a,i)$.

> *Closure under reflexivity, $\leq$-left and positivity*: trivial.
>
> *Closure under transitivity*: if $a \lhd U$ and $U \lhd W$ then $a \lhd W$. The proof is by induction on the derivation of $a \lhd U$, that is the property on which we apply induction is $P(a) \equiv U \lhd W \rightarrow a \lhd W$.
>
> > *reflexivity*: $a \lhd U$ is derived from $a \varepsilon U$. Then $a \varepsilon U$ and $U \lhd W$ give $a \lhd W$.
> >
> > *$\leq$-left*: $a \lhd U$ is derived from $a \leq b$ and $b \lhd U$. Then, by inductive hypothesis, from $b \lhd U$ and $U \lhd W$ we obtain $b \lhd W$, so that $a \lhd W$ by *$\leq$-left*.
> >
> > *$\leq$-infinity*: $a \lhd U$ is derived from $a \leq b$ and $C(b,i) \downarrow a \lhd U$. We apply the inductive hypothesis to $C(b,i) \downarrow a \lhd U$ and $U \lhd W$ to obtain $C(b,i) \downarrow a \lhd W$, from which $a \lhd W$ by *$\leq$-infinity*.

positivity: $a \lhd U$ is derived from $a \lhd U$ [$\mathsf{Pos}(a)$]. Assume $\mathsf{Pos}(a)$; then $a \lhd U$ with a shorter derivation, so $a \lhd W$ by inductive hypothesis and hence $a \lhd W$ by *positivity*.

Closure under $\leq$-right: if $a \lhd U$ and $a \lhd V$ then $a \lhd U \downarrow V$. To be able to go through the inductive steps we prove by induction a stronger claim, that is

$$\text{(stability)} \quad \frac{a \lhd U \qquad b \lhd V}{a \downarrow b \lhd U \downarrow V}$$

Then the original $\leq$*-right* is obtained from the special case in which $a = b$, since $a \varepsilon a \downarrow a$. The proof of *stability* is by induction on the derivation of $a \lhd U$.

reflexivity: $a \lhd U$ is derived from $a \varepsilon U$. The proof is by induction on the derivation of $b \lhd V$. If $b \lhd V$ is derived from $b \varepsilon V$ by *reflexivity*, then $a \varepsilon U$ and $b \varepsilon V$ give $a \downarrow b \subseteq U \downarrow V$ by definition of $\downarrow$, and hence $a \downarrow b \lhd U \downarrow V$ by *reflexivity*. In all the other cases, the proof is exactly as the corresponding steps in the main induction.

$\leq$*-left*: $a \lhd U$ is derived from $a \leq c$ and $c \lhd U$. Then by inductive hypothesis $c \downarrow b \lhd U \downarrow V$, but $a \downarrow b \subseteq c \downarrow b$ because $a \leq c$, and so $a \downarrow b \lhd U \downarrow V$ by using a bit of logic.

$\leq$*-infinity*: $a \lhd U$ is derived from $a \leq c$ and $C(c,i) \downarrow a \lhd U$. We have to prove that $a \downarrow b \lhd U \downarrow V$. Thus, let $x \varepsilon a \downarrow b$, that is $x \leq a$ and $x \leq b$. The inductive hypothesis, and a bit of logic, give $(C(c,i) \downarrow a) \downarrow b \lhd U \downarrow V$, that is $C(c,i) \downarrow (a \downarrow b) \lhd U \downarrow V$; hence, by logic, also $C(c,i) \downarrow x \lhd U \downarrow V$ since $x \varepsilon a \downarrow b$. But $x \leq a$ and $a \leq c$ give $x \leq c$, and hence $\leq$*-infinity* can be applied to obtain $x \lhd U \downarrow V$ as wished.

positivity: $a \lhd U$ is derived from $a \lhd U$ [$\mathsf{Pos}(a)$]. By the inductive hypothesis, $a \downarrow b \lhd U \downarrow V$ under the assumption $\mathsf{Pos}(a)$. Let $x \varepsilon a \downarrow b$ and $\mathsf{Pos}(x)$. Then $x \leq a$ and hence $\mathsf{Pos}(a)$ by *monotonicity on* $\leq$, so that $x \lhd U \downarrow V$ under the assumption $\mathsf{Pos}(x)$. Then by *positivity* $x \lhd U \downarrow V$ as wished.

Finally, we prove $a \lhd C(a,i)$, for any $a \in S$ and $i \in I(a)$. To this aim first note that, by *reflexivity* and $\leq$*-left*, $\downarrow C(a,i) \lhd C(a,i)$ and hence *a fortiori*, $C(a,i) \downarrow a \equiv \downarrow C(a,i) \cap \downarrow a \lhd C(a,i)$. Then $a \lhd C(a,i)$ follows by $\leq$*-infinity* since $a \leq a$.

Thus, we finished with the generation of the cover relation. To prove the second statement in the theorem we have to prove only *monotonicity* of Pos with respect to the cover that we have generated by *reflexivity*, $\leq$*-left*, $\leq$*-infinity* and *positivity*. So, let us assume that $\mathsf{Pos}(a)$ and $a \lhd U$. Then the proof is by induction on the derivation $a \lhd U$.

reflexivity: $a \lhd U$ is derived from $a \varepsilon U$. Then trivially $\mathsf{Pos}(U)$.

$\leq$-left: $a \lhd U$ is derived from $a \leq b$ and $b \lhd U$. Then, by *monotonicity on* $\leq$, we get $\mathsf{Pos}(b)$ and hence $\mathsf{Pos}(U)$ by inductive hypothesis.

$\leq$-infinity: $a \lhd U$ is derived from $a \leq c$ and $C(c,i) \downarrow a \lhd U$, for some i in $I(c)$. Then by *monotonicity on $\leq$-infinity* we obtain that $\mathsf{Pos}(C(c,i) \downarrow a)$ and hence $\mathsf{Pos}(U)$ follows by inductive hypothesis (and a bit of logic).

positivity $a \lhd U$ is derived from $a \lhd U$ $[\mathsf{Pos}(a)]$. By induction hypothesis we obtain $\mathsf{Pos}(U)$ $[\mathsf{Pos}(a)]$ but $\mathsf{Pos}(a)$ is assumed, hence $\mathsf{Pos}(U)$.

3.3 Localizing the axioms

The main advantage of the approach above is that it is possible to choose the axioms in a completely free way, that is without any condition, but the drawback is the presence of a rule *ad hoc*, namely *$\leq$-infinity*. As we showed, the aim of *$\leq$-infinity* is to be able to lift localization up to the axioms. But the axioms are contained in the rule of *infinity* itself, and this is why we had to consider its localized form *$\leq$-infinity*. So, if some axioms are given which are already localized in a suitable sense, an expectation is that the *infinity* rule will be enough. We now prove that it is so.

Definition 3.4 (Localized axiom-set) *Let I and C be an axiom-set. Then we say that it is* localized *if, for any $a \leq c$ and $i \in I(c)$, there exists $j \in I(a)$ such $C(a,j) \subseteq a \downarrow C(c,i)$.*

Then, we can prove the following proposition.

Proposition 3.5 *Provided the axiom-set is localized, an infinitary relation generated by using* reflexivity*,* $\leq$-left *and* infinity *satisfies* $\leq$-infinity*.*

Proof. Let us suppose that $a \leq c$ and $C(c,i) \downarrow a \lhd U$. Then, by assumption, there exists $j \in I(a)$ such that $C(a,j) \subseteq a \downarrow C(c,i)$. So $C(a,j) \lhd U$ and hence $a \lhd U$ by *infinity*.

Thus, it is to be expected that it is possible to generate a formal topology also by using *infinity* instead of *$\leq$-infinity*, provided that the axiom-set is localized. In fact, we can prove the following theorem.

Theorem 3.6 *Let S be any set, $\leq$ any pre-order on S and let $I(a)$* set *$[a : S]$ and $C(a,i) \subseteq S$ $[a : S, i : I(a)]$ be a localized axiom-set. Assume in addition that a predicate $\mathsf{Pos}(a)$* set *$[a : S]$ is given which satisfies* monotonicity on $\leq$ *and* monotonicity on infinity*, that is, if $\mathsf{Pos}(a)$ and $i \in I(a)$, then $\mathsf{Pos}(C(a,i))$. Let $\lhd$ be the infinitary relation generated by* reflexivity*,* $\leq$-left*,* infinity *and* positivity*. Then $\mathcal{A} \equiv (S, \leq, \lhd, \mathsf{Pos})$ is a $\leq$-formal topology and $\lhd$ is the least among the covers $\lhd'$ satisfying $a \lhd' C(a,i)$ $[a : S, i : I(a)]$ and making $(S, \leq, \lhd', \mathsf{Pos})$ a formal topology.*

Proof. The proof is almost exactly the same as in theorem 3.3 except the inductive steps concerning the usage of $\leq$-*infinity* that we modify as follows:

> *transitivity*: $a \lhd U$ is derived from $C(a,i) \lhd U$ by *infinity*. Then, supposing $U \lhd V$, $C(a,i) \lhd V$ follows by inductive hypothesis and hence $a \lhd V$ by *infinity*.
>
> *stability*: $a \lhd U$ is derived by $C(a,i) \lhd U$ by *infinity*. We want to prove that $a \downarrow b \lhd U \downarrow V$. Then let $x \varepsilon a \downarrow b$, that is $x \leq a$ and $x \leq b$. The inductive hypothesis, and a bit of logic, give $C(a,i) \downarrow b \lhd U \downarrow V$, and hence also $C(a,i) \downarrow x \lhd U \downarrow V$ since $x \leq b$ gives $\downarrow x \subseteq \downarrow b$. Since the axiom-set is localized and we assumed that $x \leq a$, there exists $j \in I(x)$ such that $C(x,j) \subseteq x \downarrow C(a,i)$ and hence $C(x,j) \lhd U \downarrow V$ which by *infinity* gives $x \lhd U \downarrow V$ as wished.

Also the axioms can be proved by using *infinity* instead of $\leq$-*infinity*, and with a simpler proof. In fact, for any $a \in S$ and any $i \in I(a)$, $C(a,i) \lhd C(a,i)$ by *reflexivity* and hence $a \lhd C(a,i)$ by *infinity*.

Finally, we must give the inductive step to prove *monotonicity* when the rule applied to prove $a \lhd U$ is *infinity*. Hence we know $C(a,i) \lhd U$ for some $i \in I(a)$; so by *monotonicity on infinity* $\mathsf{Pos}(a)$ gives $\mathsf{Pos}(C(a,i))$ and hence $\mathsf{Pos}(U)$ by inductive hypothesis.

Given any axiom-set I, C we can always provide with a *new* axiom-set J, D which is localized and generates the same cover relation. In fact, it is possible to prove the following theorem.

Theorem 3.7 *Let $I(a)$ set $[a : S]$ and $C(a,i) \subseteq S$ $[a : S, i : I(a)]$ be an axiom-set and define a new axiom-set by putting*

$$\begin{array}{lcl} J(a) & \equiv & \{\langle c,k\rangle \mid (a \leq c) \ \&\ k \in I(c)\} \\ D(a,\langle c,k\rangle) & \equiv & a \downarrow C(c,k) \end{array}$$

Then, if $a \leq c$ and $i \in J(c)$ there exists $j \in J(a)$ such that $D(a,j) \subseteq a \downarrow D(c,i)$, that is, the new axiom-set is localized. Moreover, the axiom-set J and D generates the same cover relation than the axiom-set I and C.

Its proof is almost immediate if one observe that, if $a \leq c$ and $\langle d,k\rangle \in J(c)$ then $\langle d,k\rangle \in J(a)$ and $D(a,\langle d,k\rangle) = a \downarrow C(d,k) = a \downarrow c \downarrow C(d,k) = a \downarrow D(c,\langle d,k\rangle)$.

4 Some examples and one counter-example

This section is devoted to show some examples of application of the general technique that we exploited in the previous sections to inductively generate formal topologies. We will see that most of the standard topological spaces and topological constructions can indeed be inductively generated and hence that almost nothing is lost by restricting to consider only inductively generated formal topologies. Anyhow we will also show a remarkable example of formal topology which cannot be inductively defined: this fact suggests to keep the very definition of the notion of formal topology in its full generality instead to restrict to consider only inductively generated formal topologies.

4.1 The Cantor space

The first example that we want to show is the formal topology over binary trees, that is, Cantor space. To define such a formal topology we will use the set $\mathbf{2}^*$ of binary words, that is, finite sequences of 0 and 1. To obtain a $\sqsubseteq$-formal topology from the set $\mathbf{2}^*$ we use the order relation $\sqsubseteq$ such that, for any two words σ and σ', $\sigma \sqsubseteq \sigma'$ holds if and only if σ' is an initial segment of σ. relation. The intended meaning of such an order relation can be understood if one thinks of a word as a *partial* information over an infinite sequence, which corresponds intuitively to a *classical* function from the set N of natural number into $\mathbf{2}$; thus a longer word is a more precise information on the infinite sequence and hence there are *less* words that contains it than any of its initial segments. Finally the empty word, which is contained in all words, gives no information at all and is contained in all of the infinite sequences. In formal topology a representation of such infinite sequences can be obtained by using the collection of the formal points. Given any formal topology $(S, \leq, \lhd, \mathsf{Pos})$, a *formal point* is any non-empty subset α of S such that, for any $a, b \in S$ and $U \subseteq S$, both if $a\varepsilon\alpha$ and $b\varepsilon\alpha$ then there exists $c\varepsilon\alpha$ such that $c \lhd a$ and $c \lhd b$, and if $a\varepsilon\alpha$ and $a \lhd U$ then there exists $u\varepsilon U$ such that $u\varepsilon\alpha$. The collection of all the formal points will be indicated by $\mathsf{Pt}(S)$. Then an infinite sequence f can be identified with the formal point whose elements are all the words σ such that for any natural number x, smaller then the length of the word σ, $f(x)$ is equal to $\sigma[x]$, where $\sigma[x]$ is the value in the place x of σ.

With the same intuitive reading it is also easy to understand that $\sigma \lhd U$ should means that the words which contain σ are all contained in the words that contain at least one of the words in U. But in order to inductively generate this cover relation we have to state suitable axioms for it. Here we simply require that the word σ is covered by all its one-step successors, that is the only form of axiom is

$$\sigma \lhd \{\sigma 0, \sigma 1\}$$

which is clearly an axiom-set. It is easy to prove that this axiom-set is localized and hence the simple *infinity* rule is sufficient to generate this formal topology. Finally the positivity predicate is completely trivial since any word is positive, that is, we put

$$\mathsf{Pos}(\sigma) \equiv \mathsf{true}$$

For more information about this formal topology and the collection of its formal points see [Val99].

It can be useful to show directly an axiom-set for the full cover relation of this simple formal topology without making a reference to the proof of theorem

3.2. First note that

$$\sigma \lhd U \text{ if and only if } (\exists n \in \mathsf{N})(\forall \sigma' \in \mathbf{2}^*(n))(\exists u \varepsilon U)\ \sigma\sigma' \sqsubseteq u$$

where $\mathbf{2}^*(n)$ is the set of the sequence of 0 and 1 of length n and $\sigma\sigma'$ is the concatenation of the two words σ and σ'. In fact, one direction can be proved by induction on the natural number n which is assumed to exist while the other can be proved by induction on the length of the derivation of $\sigma \lhd U$.

We can now use a choice principle, which is valid in constructive type theory, and obtain that

$$\begin{array}{l} \sigma \lhd U \text{ if and only if} \\ \quad (\exists n \in \mathsf{N})(\exists f \in \mathbf{2}^*(n) \to \mathbf{2}^*)(\forall \sigma' \in \mathbf{2}^*(n))\ \sigma\sigma' \sqsubseteq f(\sigma')\ \&\ f(\sigma')\varepsilon U \end{array}$$

Then, by putting

$$\begin{array}{rcl} I(\sigma) & \equiv & \{\langle n, f\rangle |\ n \in \mathsf{N}, f \in \mathbf{2}^*(n) \to \mathbf{2}^*, (\forall \sigma' \in \mathbf{2}^*(n))\ \sigma\sigma' \sqsubseteq f(\sigma')\} \\ C(\sigma, \langle n, f\rangle) & \equiv & \mathsf{Im}[f] \end{array}$$

where $\mathsf{Im}[f] \equiv \{\sigma \in \mathbf{2}^* |\ (\exists \sigma' \in \mathbf{2}^*(n))\ \sigma = f(\sigma')\}$ is the image of the function f, we obtain

$$\sigma \lhd U \text{ if and only if } (\exists \langle n, f\rangle \in I(\sigma))\ C(\sigma, \langle n, f\rangle) \subseteq U$$

that is, I and C is an axiom-set for the Cantor formal topology.

4.2 The real numbers

Our second example of a formal topology which can be inductively generated is the formal topology of real numbers. We can identify a real number with a suitable collection of open intervals on the rational line, that is, the collection of all the intervals which contain such a real number (see for instance [Joh82, Ver86]). Thus, a real number can be identified with a formal point of a suitable $\leq$-formal topology whose basic opens are the open intervals of the rational line $\mathcal{Q}$. We obtain such a $\leq$-formal topology $(S, \leq, \lhd, \mathsf{Pos})$ by putting

$$S \equiv \mathcal{Q} \times \mathcal{Q}$$

The order relation $\leq$ among these intervals is then simply defined by using the standard order relation $\leq$ between rational numbers and putting

$$(p, q) \leq (r, s) \equiv (r \leq p)\ \&\ (q \leq s)$$

The intended meaning is that $(p, q) \leq (r, s)$ when the interval (p, q) is contained in the interval (r, s).

Let us note now that the interval (p, q) is meant to contain some point only if $p < q$ and hence the positivity predicate can be defined by putting

$$\mathsf{Pos}((p, q)) \equiv p < q$$

We can state now the axioms that we need:

$$(p, q) \lhd \{(p, s), (r, q)\} \text{ for } r < s$$

$$(p, q) \lhd \{(r, s) |\ p < r < s < q\}$$

The geometrical meaning of these axioms should be clear. In fact, let us suppose that (p, q) is inhabited, then any axiom in the first family of axioms states that an interval (p, q) is covered by any pair of overlapping intervals which are contained in (p, q) provided one covers the left part and the other the right part of (p, q). And the second axiom states that an interval is covered by the collection of all the intervals that it contains properly. These axioms form an axiom-set since for any couple $(p, q) \in \mathcal{Q} \times \mathcal{Q}$ we can define $I((p, q))$ to be the set obtained by adding one element $*$ to the set of the ordered couple of rational numbers. In fact, we can now use any ordered couple (r, s) of rational numbers such that $r < s$ to index one of the axioms of the first family and the element $*$ to index the only axiom of the second kind.

It is worth noting that also in this case the axiom-set that we are considering is localized and hence this formal topology can be generated by using *infinity* instead that the more complex $\leq$-*infinity*. This observation is useful to understand that to inductively generate this formal topology one can equivalently use the following two rules

$$\frac{r < s \quad \{(p, s), (r, q)\} \lhd U}{(p, q) \lhd U} \qquad \frac{\{(r, s) |\ p < r < s < q\} \lhd U}{(p, q) \lhd U}$$

Note that it can be shown that to prove $(p, q) \lhd U$ one needs to use the second rule at most once at the end of the proof.

4.3 Cartesian product revisited

In this example we will go back to the problem of a predicative definition of the cartesian product of formal spaces. Indeed, in section 1 we observed that we know no predicative way, apart inductive generation, to construct the cartesian product of formal spaces. After the previous sections we know that we can inductively generate a formal topology provided it has an axiom-set. Thus, we can present now our solution to the problem of the predicative definition of the cartesian product of formal topologies. In fact, if $\mathcal{S}$ and $\mathcal{T}$ are two formal topologies with an axiom-set then their cartesian product, like in the definition at the end of section 1, is again a formal topology which has an axiom-set. Indeed, it is sufficient to observe that if the axiom-set for the formal topology $\mathcal{S}$ is the family of sets $I(a)$ set $[a : S]$ together with the family of subsets $C(a, i) \subseteq S$ $[a : S, i : I(a)]$ and the axiom-set for the topology $\mathcal{T}$ is the family of sets $J(b)$ set $[b : T]$ together with the family of subsets $D(b, j) \subseteq T$ $[b : T, j : J(b)]$ then the axiom-set for the cartesian product of $\mathcal{S}$ and $\mathcal{T}$ is the following

$$(a, b) \lhd C(a, i) \times b \ [a : S, b : T, i : I(a)]$$
$$(a, b) \lhd a \times D(b, j) \ [a : S, b : T, j : J(b)]$$

where

$$\begin{array}{rcl} C(a,i) \times b & \equiv & \{(z,b) \in S \times T \mid z \varepsilon C(a,i)\} \\ a \times D(b,j) & \equiv & \{(a,w) \in S \times T \mid w \varepsilon D(b,j)\} \end{array}$$

It is clear that such an axiom-set can be indexed by using the following family of sets:

$$K((a,b)) \equiv (I(a) \times T) + (S \times J(b))$$

where $A + B$ is the disjoint union of the two sets A and B.

Note that the above definition of product of $\mathcal{S}$ and $\mathcal{T}$ works only if $\mathcal{S}$ and $\mathcal{T}$ have an axiom-set. Since not all the formal topologies have an axiom-set (see section 4.7) the problem of an unrestricted predicative definition of product of formal topologies is still open.

4.4 All representable topologies have an axiom-set

In the previous sections we have given single examples of formal topologies which have an axiom-set and this might suggest the idea that just a few, although important, formal topologies have an axiom-set. Actually one can show that large classes of formal topologies have an axiom-set. Here we do this for representable formal topologies, in the next section for unitary and finitary formal topologies.[2]

We say that a formal topology $\mathcal{A} \equiv (S, \lhd, \mathsf{Pos})$ is *representable* if there exist a set X and a proposition $x \Vdash a$ prop $[x : X, a : S]$ such that

$$\begin{array}{lll} a \lhd U & \text{if and only if} & (\forall x \in X)\ (x \Vdash a \rightarrow (\exists b \in S)\ x \Vdash b \ \&\ b \varepsilon U) \\ \mathsf{Pos}(a) & \text{if and only if} & (\exists x \in X)\ x \Vdash a \end{array}$$

That is, a formal topology is representable when it is the formal part of a concrete topological space.

It is interesting to note that when a formal topology $\mathcal{S}$ is representable the set X of concrete points can be embedded into the collection $\mathsf{Pt}(S)$ of the formal points of S. In fact, it is possible to associate to the concrete point x in X the formal point $\alpha_x \equiv \{a \in S \mid x \Vdash a\}$.

If we are dealing with a representable formal topology, we can apply a choice principle, which is valid in constructive type theory, and obtain that

$$a \lhd U$$

if and only if

$$(\exists f \in \{g \in \mathsf{ext}(a) \rightarrow S \mid (\forall x \varepsilon \mathsf{ext}(a))\ x \Vdash g(x)\})(\forall x \varepsilon \mathsf{ext}(a))\ f(x) \varepsilon U$$

that is, the function f takes any point x contained in $\mathsf{ext}(a)$ and finds an element b in S such that x is contained in $\mathsf{ext}(b)$ and b is an element of the subset U. Hence, we can put

$$\begin{array}{rcl} I(a) & \equiv & \{g \in \mathsf{ext}(a) \rightarrow S \mid (\forall x \varepsilon \mathsf{ext}(a))\ x \Vdash g(x)\} \\ C(a,f) & \equiv & \mathsf{Im}[f] \end{array}$$

[2] The idea for the results in this section comes us from a conversation with Per Martin-Löf.

where $\mathsf{Im}[f]$ is the image of f.

Then we obtain

$$a \lhd U \text{ if and only if } (\exists f \in I(a))\ C(a,f) \subseteq U$$

that is, I and C is an axiom-set for $\lhd$. In fact, if $a \lhd U$ then by the above stated principle of choice we can find the suitable function in $I(a)$. On the other hand, if $(\exists f \in I(a))\ C(a,f) \subseteq U$ then we have a function such that for any point x contained in $\mathsf{ext}(a)$ gives an element b of S such that $x \varepsilon \mathsf{ext}(b)$; thus, since the topology is representable, $a \lhd \mathsf{Im}[f]$ holds; but also $\mathsf{Im}[f] \subseteq U$ holds and hence $a \lhd U$ follows by *transitivity*.

Note that, as a consequence of the fact that all of the representable formal topologies can be inductively generated and of the existence of a formal topology which cannot be generated, that we will present in section 4.7, we know that not all formal topologies are representable. Moreover, it is interesting to note that there are formal topologies which can be inductively generated but are not representable. For instance, in section 4.1 we proved that Cantor space is a formal topology inductively generated; on the other hand it cannot be representable, at least if we want to represent it by using the most natural choice, that is by using like set X the set $\mathbf{2}^{\mathsf{N}}$ of the functions from the natural numbers into the two elements set $\mathbf{2}$. Indeed in this case the most natural definition of the forcing relation would be

$$x \Vdash \sigma \equiv (\forall n < \mathsf{len}(\sigma))\ f(x) = \sigma[x]$$

where $\mathsf{len}(\sigma)$ is the length of the sequence σ and $\sigma[x]$ is the value in the x-th position of the sequence σ. Now, to state that X and $\Vdash$ can be used to represent the Cantor space is equivalent to the Brouwer's fan theorem and hence it is not recursively valid (see [JT84] and [FG82]).

4.5 A space whose collection of points is not a set

Let us give an example of a formal space whose collection of points cannot be indexed by a set. Let us consider the set $\mathbf{2}$ and the order relation defined by putting $0 \leq 0$, $0 \leq 1$ and $1 \leq 1$. Then we can obtain a cover relation by putting, for any $a \in \mathbf{2}$ and $U \subseteq \mathbf{2}$,

$$a \lhd U \equiv (\exists u \varepsilon U)\ a \leq u$$

In fact, it is immediate to check that the relation $\lhd$ satisfies *reflexivity*, *transitivity*, *$\leq$-left* and *$\leq$-right*. This cover relation is the minimal cover relation that one can define to obtain a $\leq$-formal topology over $(\mathbf{2}, \leq)$, that is, it is the cover of the formal topology generated on $(\mathbf{2}, \leq)$ by using no axiom, that is when in the axiom-set, for any $a \in \mathbf{2}$, the set $I(a)$ is empty. Note that the axiom-set for this formal topology is trivially localized.

Moreover, it is not difficult to check that a subset α of $\mathbf{2}$ is a formal point of this formal topology if and only if both $1 \varepsilon \alpha$ and whenever $a \varepsilon \alpha$ and $a \leq b$ then also $b\varepsilon\alpha$. Thus, for any closed proposition A, $\alpha_A \equiv \{x \in \mathbf{2} |\ A \vee (x =_2 1)\}$ is a point. Now, note that $\alpha_A = \alpha_B$ if and only if $A \leftrightarrow B$. Thus, there is a bijective correspondence between the collection of formal points of this formal topology and the collection of the closed propositions, because we can associate to any formal point α the closed proposition $A_\alpha \equiv 0\varepsilon\alpha$ and in this way we obtain that $\alpha = \alpha_{A_\alpha}$. But then if it would be possible to index by a set the collection of the formal points of this formal topology then it could as well be possible to index by a set the collection of the closed propositions, that is the collection of the subsets of the one element set, and it is not expected that such a collection can be indexed by any set.

4.6 Unitary and finitary formal topologies

In this section we will show that also two other classes of formal topologies have an axiom-set, namely Scott's and the Stone's formal topologies.

Let us first analyze the case of Stone's, alias finitary, formal topologies since they are technically simpler than Scott's topologies because they do not require a detailed treatment of the positivity predicate. On the other hand, after Stone's topologies will be understood, in order to deal with Scott's topologies we will have only to specialize the ideas that we used for the former to the latter and thus we will be able to work out the details which will allow us to deal with the positivity predicate.

Let S be any set and $\mathcal{P}_\omega(S)$ be the set of the finite subsets of S[3].

Then we can give the following definition.

Definition 4.1 *A formal topology* $\mathcal{A} \equiv (S, \lhd, \mathsf{Pos})$ *is* finitary *if* $a \lhd U$ *if and only if* $(\exists U_0 \in \mathcal{P}_\omega(S))\ (U_0 \subseteq U\ \&\ a \lhd U_0)$.

Given any finitary formal topology $(S, \lhd, \mathsf{Pos})$ we will call *trace* of $\mathcal{S}$ the relation $\mathsf{Tr}(a, U_0)$, where a is any element in S and U_0 is any finite subset of S, which holds if and only if $a \lhd U_0$.

Then, supposing $\mathcal{A} \equiv (S, \lhd, \mathsf{Pos})$ is a finitary formal topology whose trace is definible by the proposition $\mathsf{Tr}(a, U_0)$, the axiom-set that one needs in order to inductively generate $\mathcal{A}$ is the following

$$\begin{array}{rcl} I(a) & \equiv & \{U_0 \in \mathcal{P}_\omega(S) |\ \mathsf{Tr}(a, U_0)\} \\ C(a, U_0) & \equiv & U_0 \end{array}$$

It is now possible to prove that

$$a \lhd U \text{ if and only if } (\exists U_0 \in I(a))\ C(a, U_0) \subseteq U$$

Both directions are trivial. Suppose that $(\exists U_0 \in I(a))\ C(a, U_0) \subseteq U$ holds; then, $U_0 \in I(a)$ yields $a \lhd U_0$, that is, $a \lhd C(a, U_0)$ and hence $a \lhd U$ follows by

[3] We will not commit here with any particular implementation of this set to avoid to deal with the problems that would arise (see for instance [Mai99]). In any case all of what we are doing here can be formalize in Martin-Löf's type theory by using the type $\mathsf{List}(S)$ of the lists of elements of S [NPS90].

transitivity, since $C(a, U_0) \subseteq U$. To prove the other implication, let us assume that $a \lhd U$, that is, $(\exists U_0 \in \mathcal{P}_\omega(S))\ (U_0 \subseteq U\ \&\ a \lhd U_0)$ since the cover is finitary. Then $U_0 \in \mathcal{P}_\omega(S)$ and $\mathsf{Tr}(a, U_0)$, that is $U_0 \in I(a)$, and $U_0 \subseteq U$, that is $C(a, U_0) \subseteq U$.

Let us analyze now the case of Scott, alias unitary, formal topologies. The definition of unitary formal topology is the following:

Definition 4.2 *The formal topology* $\mathcal{A} \equiv (S, \lhd, \mathsf{Pos})$ *is* unitary *if* $a \lhd U$ *if and only if* $\mathsf{Pos}(a) \to (\exists b \in S)\ (b \varepsilon U\ \&\ a \lhd b)$.

Also in this case there is a natural notion of *trace* Tr of the cover relation, that is, $\mathsf{Tr}(a, b)$, for any $a, b \in S$, is a relation which holds if and only if $a \lhd b$.

The main novelty of unitary topologies with respect to the previous case of finitary topologies is the presence of the assumption on the positivity of a in the condition on $a \lhd U$ which defines them, which is necessary to avoid reasoning by case, according to the positivity of a. This assumption will force us to change all of the previous definitions to adapt them to this new setting. In particular it will be necessary to deal with the proofs of the proposition $\mathsf{Pos}(a)$. This is the reason why, following the *propositions as sets* tradition, in the next proofs we will write $x \in \mathsf{Pos}(a)$ to mean that x is a proof of $\mathsf{Pos}(a)$.

Suppose $\mathcal{A} \equiv (S, \lhd, \mathsf{Pos})$ is a unitary formal topology and that its trace is defined by the proposition $\mathsf{Tr}(a, b)\ [a : S, b : S]$; then we obtain an axiom-set for $\mathcal{A}$ by putting

$$\begin{array}{rcl} I(a) & \equiv & \{g \in \mathsf{Pos}(a) \to S \mid (\forall x \in \mathsf{Pos}(a))\ \mathsf{Tr}(a, g(x))\} \\ C(a, f) & \equiv & \mathsf{Im}[f] \end{array}$$

Similarly to the previous case with finitary topologies, the intended meaning of the set $I(a)$ is to denote the set whose elements are all the elements of S which cover a but we had to move to this more complex definition because the set $\{b \in S \mid a \lhd b\}$, which is equal to the set $\{b \in S \mid \mathsf{Pos}(a) \to a \lhd b\}$ because of the *positivity* condition, is only classically equivalent to the set $I(a)$ in our definition and in the next proofs we will need exactly $I(a)$. The intended meaning of the definition of the family of subset $C(a, f)$ is to obtain, according to the fact that a is a positive element of S or not, a subset which is a singleton subset of S, whose only element is an element which covers a, or the empty subset; the way the subset $C(a, f)$ is defined, which can look a bit strange at a first sight, was chosen to guarantee the property above without requiring decidability of the predicate Pos.

It is now possible to show that

$$a \lhd U \text{ if and only if } (\exists f \in I(a))\ C(a, f) \subseteq U$$

that is, any unitary formal topology with a definable trace has an axiom-set, and hence can be inductively generated.

Let us first suppose that $(\exists f \in I(a))\ C(a, f) \subseteq U$ holds. Then $f \in I(a)$ and hence, assuming x is a proof of $\mathsf{Pos}(a)$, we can deduce that $a \lhd f(x)$ and hence also $a \lhd C(a,f)$ and finally $a \lhd U$ by *transitivity*, since $f(x)\varepsilon\mathsf{Im}[f] \equiv C(a,f)$ and $C(a,f) \subseteq U$. Then $a \lhd U$ without any assumption follows since the assumption $\mathsf{Pos}(a)$ can be discharged by *positivity*.

Now, let us suppose that $a \lhd U$. Then $\mathsf{Pos}(a) \to (\exists b \in S)\ (b\varepsilon U\ \&\ a \lhd b)$ follows because the formal topology is unitary. But then also

$$(\forall x \in \mathsf{Pos}(a))(\exists b \in S)\ (b\varepsilon U\ \&\ a \lhd b)$$

holds, and hence we can use a valid choice principle and obtain that

$$(\exists f \in \mathsf{Pos}(a) \to S)(\forall x \in \mathsf{Pos}(a))\ f(x)\varepsilon U\ \&\ a \lhd f(x)$$

Thus we obtain both that $f \in \mathsf{Pos}(a) \to S$ and $(\forall x \in \mathsf{Pos}(a))\ \mathsf{Tr}(a, f(x))$, that is $f \in I(a)$, and $(\forall x \in \mathsf{Pos}(a))\ f(x)\varepsilon U$, that is $C(a,f) \equiv \mathsf{Im}[f] \subseteq U$.

4.7 A covering which cannot be inductively defined

We want to finish this section by showing that not all of the formal topologies can be obtained by inductive generation. To this aim let us introduce the notion of Dedekind-MacNeille cover. Let S be a set and $\leq$ be an order relation on the elements of S. Then, inspired by the Dedekind construction of the completion of the rational line, we can define an infinitary relation $\lhd_{DM}$ over S by putting, for any $a \in S$ and any $U \subseteq S$,

$$a \lhd_{DM} U \equiv a\varepsilon \bigcap_{U \subseteq \downarrow y} \downarrow y$$

where $\downarrow y \equiv \{x \in S|\ x \leq y\}$. It is not difficult to prove that this relation satisfies *reflexivity*, *transitivity* and $\leq$*-left* while to prove that also $\leq$*-right* holds some more conditions on the order relation are required like, for instance, the possibility of defining both an infimum operation between two elements of S and its adjoint, namely implication (see [Sam89]). We will call such a cover the Dedekind-MacNeille cover over S.

We want now to show that there are examples of a Dedekind-MacNeille cover which do not have an axiom-set. To this aim it is convenient to use an equivalent formulation of the notion of axiom-set.

Definition 4.3 *A* bunch *of axioms for a cover relation on a set S is a family of sets $B(a)$* set *$[a : S]$ and a family of sets $C(a,b)$* set *$[a : S, b : B(a)]$ together with a function $d(a,b,c) \in S\ [a : S, b : B(a), c : C(a,b)]$ whose intended meaning is that, for all $a \in S$ and $b \in B(a)$, $a \lhd \mathsf{Im}[\lambda x.\ d(a,b,x)]$ where $\mathsf{Im}[\lambda x.\ d(a,b,x)] \equiv \{x \in S|\ (\exists c \in C(a,b))\ x =_S d(a,b,c)\}$.*

We know that a cover relation $\lhd$ on a set S is set-based when there is an axiom-set $I(a)$ set $[a : S]$ and $K(a,b) \subseteq S\ [a : S, b : I(a)]$ such that $a \lhd U$ if and only if $(\exists b \in I(a))\ K(a,b) \subseteq U$. A similar condition can be used with a bunch of axioms. We say that a cover relation is set-based if $a \lhd U$ if and only if $(\exists b \in B(a))(\forall c \in C(a,b))\ d(a,b,c)\varepsilon U$.

Then, it is possible to prove that the notion of bunch of axioms is equivalent to the notion of axiom-set. In fact, suppose $a \in S$ and $U \subseteq S$ and assume to have a bunch of axioms $B(a)$ set $[a : S]$, $C(a,b)$ set $[a : S, b : B(a)]$ and $d(a,b,c) \in S$ $[a : S, b : B(a), c : C(a,b)]$. Then we can define an equivalent axiom-set by putting

$$\begin{array}{rcl} I(a) & \equiv & B(a) \\ K(a,b) & \equiv & \mathsf{Im}[\lambda x.d(a,b,x)] \end{array}$$

In fact, it is now possible to prove that if $(\exists b \in B(a))(\forall c \in C(a,b))\ d(a,b,c)\varepsilon U$ then $(\exists b \in I(a))\ K(a,b) \subseteq U$. In fact by using the same element b that we know to exist in $I(a) \equiv B(a)$ from the hypothesis, we have just to show that, for all $x \in S$, if $x \varepsilon K(a,b)$ then $x \varepsilon U$. But $x \varepsilon K(a,b)$ means that there is an element $c \in C(a,b))$ such that $x =_S d(a,b,c)$ and then the hypothesis yields that $d(a,b,c)\varepsilon U$ and hence $x\varepsilon U$.

On the other hand, if we start with an axiom-set $I(a)$ set $[a : S]$ and $K(a,b) \subseteq S$ $[a : S, b : I(a)]$ then we can define an equivalent bunch of axioms by putting

$$\begin{array}{rcl} B(a) & \equiv & I(a) \\ C(a,b) & \equiv & \Sigma(S, K(a,b)) \\ d(a,b,c) & \equiv & \mathsf{fst}(c) \end{array}$$

Indeed, it is now possible to prove that, if $(\exists b \in I(a))\ K(a,b) \subseteq U$, then $(\exists b \in B(a))(\forall c \in C(a,b))\ d(a,b,c)\varepsilon U$. In fact by using the same element b that we know to exist in $B(a) \equiv I(a)$ from the hypothesis, we have to show that $(\forall c \in C(a,b))\ d(a,b,c)\varepsilon U$. Thus, let us assume that $c \in \Sigma(S, K(a,b))$. Then $\mathsf{snd}(c)$ is a proof that $\mathsf{fst}(c)\varepsilon K(a,b)$ and hence the hypothesis shows that it belongs also to U; thus $d(a,b,c)$ belongs to U since $d(a,b,c) \equiv \mathsf{fst}(c)$.

Now, let $S \in \mathsf{Set}$, $B(x) \in \mathsf{Set}$ $[x \in S]$, $C(x,y) \in \mathsf{Set}$ $[x \in S,\ y \in B(x)]$, $d(x,y,z) \in S$ $[x \in S,\ y \in B(x),\ z \in C(x,y)]$ be given.

We let $\prec \in S \to (S \to \mathsf{Set}) \to \mathsf{Set}$ denote the infinitary relation inductively defined by the rules

$$\frac{a \epsilon U}{a \prec U}$$

$$\frac{b \in B(a) \qquad (\forall z \in C(a,b))\,(d(a,b,z) \prec U)}{a \prec U}$$

Intuitively, we can think of $B(a)$ as the set of rules by which we can conclude $a \prec U$ for $a \in S$. $C(a,b)$ is then the set of indexes for the premisses of the rule: for each z in $C(a,b)$, $d(a,b,z) \prec U$ is a premiss.

In general, $\prec$ is not a covering relation. However, given a partial order $\leq$, the rule

$$\frac{a' \prec U \qquad a \leq a'}{a \prec U}$$

is easily expressed by the schema above; hence, every inductively defined covering relation can be obtained in this way.

For arbitrary $a \in S$, define the set $\mathsf{Tree}(a)$ of trees with root a by induction:

$$\mathsf{leaf\_tree}(a) \in \mathsf{Tree}(a)$$

$$\frac{b \in B(a) \qquad f \in (z \in C(a,b))\mathsf{Tree}(d(a,b,z))}{\mathsf{branch\_tree}(a,b,f) \in \mathsf{Tree}(a)}$$

The set of branches $\mathsf{Branch}(a,t)$ of a tree $t \in \mathsf{Tree}(a)$ is defined by

$$\begin{cases} \mathsf{Branch}(a, \mathsf{leaf\_tree}(a)) & = \mathsf{N}_1 \\ \mathsf{Branch}(a, \mathsf{branch\_tree}(a,y,f)) & = (\Sigma z \in C(a,y))(\mathsf{Branch}(d(a,y,z), f(z))) \end{cases}$$

where N_1 is the unit set.

We also need the leaf of a branch u of a tree t:

$$\begin{cases} \mathsf{lf\_br}(a, \mathsf{leaf\_tree}(a), u) & = a \\ \mathsf{lf\_br}(a, \mathsf{branch\_tree}(a,y,f), (v,w)) & = \mathsf{lf\_br}(d(a,y,v), f(v), w) \end{cases}$$

We can now give an explicit description of covers by putting

$$\mathsf{cover}(U,a,t) \ = \ (u \in \mathsf{Branch}(a,t))\, U(\mathsf{lf\_br}(a,t,u))$$

Theorem 4.4 *$a \prec U$ holds if and only if there exists $t \in \mathsf{Tree}(a)$ such that $\mathsf{cover}(U,a,t)$ holds.*

Proof. The tree t is easily constructed by induction on the derivation of $a \prec U$. That $\mathsf{cover}(U,a,t)$ implies $a \prec U$ is straightforwardly proved by induction t.

We now come to the topology which covering relation cannot be inductively defined. The base of the topology is $\mathsf{Boole} = \{0,1\}$ with the natural ordering $\leq$ and with the Dedekind-MacNeille cover

$$x \lhd U \ \Leftrightarrow \ (\forall y)[(\forall u \,\epsilon\, U)\, u \leq y] \Rightarrow x \leq y$$

This covering relation can be redefined as

$$x \lhd U \ \Leftrightarrow \ x \,\epsilon\, U''$$

where $V' = \{x \in \mathsf{Boole} \mid (\forall u \,\epsilon\, V)\, xu = 0\}$ and the product is propositional conjunction. Indeed, by taking $z = \neg y$ in $(\forall y)[(\forall u \,\epsilon\, U)\, u \leq y] \Rightarrow x \leq y$ we see that this proposition is equivalent to

$$\forall z[(\forall u \,\epsilon\, U)uz = 0] \Rightarrow xz = 0$$

that is, $U' \subset \{x\}'$, which is equivalent to $x \,\epsilon\, U''$.

Assume that $\lhd$ can be inductively defined. By theorem 4.4, we then have a tree structure $\mathsf{Tree}(x) \in \mathsf{Set}\ [x \in \mathsf{Boole}]$, $\mathsf{Branch}(x,y) \in \mathsf{Set}\ [x \in \mathsf{Boole},\ y \in \mathsf{Tree}(x)]$ such that

$$a \,\epsilon\, U'' \ \Leftrightarrow \ (\exists t \in \mathsf{Tree}(a))(\forall z \in \mathsf{Branch}(a,t))U(\mathsf{lf\_br}(a,t,z)) \tag{1}$$

for all $U \in (x \in \mathsf{Boole})\,\mathsf{Set}$ because, as we have seen above, $a \lhd U \;\Leftrightarrow\; a \,\epsilon\, U''$. Let V be an arbitrary subset of Boole, and let P_1 denote the proposition $1 \,\epsilon\, V$. We then have $V' \;=\; \{0\} \cup \{1 \mid \neg P_1\}$. Hence

$$V'' \;=\; \{0\} \cup \{1 \mid \neg\neg P_1\} \tag{2}$$

Let P be an arbitrary proposition and define U_P by

$$x \,\epsilon\, U_P \;\Leftrightarrow\; [x = 0 \vee (x = 1 \,\&\, P)]$$

By (1) we have

$$1 \,\epsilon\, U''_P \;\Leftrightarrow\; (\exists t \in \mathsf{Tree}(1))(\forall z \in \mathsf{Branch}(1,t))U_P(\mathsf{lf\_br}(1,t,z)) \tag{3}$$

Since $1 \,\epsilon\, U_P \Leftrightarrow P$ we get from (2) that $1 \,\epsilon\, U''_P \Leftrightarrow \neg\neg P$; hence

$$\neg\neg P \;\Leftrightarrow\; (\exists t \in \mathsf{Tree}(1))(\forall z \in \mathsf{Branch}(1,t))U_P(\mathsf{lf\_br}(1,t,z)) \tag{4}$$

We claim that (4) is not valid in the ω-model [Hof97]. Indeed, if we take $P \;=\; Q \vee \neg Q$, (4) implies

$$(\exists t \in \mathsf{Tree}(1))(\forall z \in \mathsf{Branch}(1,t))U_{Q\vee\neg Q}(\mathsf{lf\_br}(1,t,z))$$

because $\neg\neg(Q \vee \neg Q)$ is logically true. Hence, by the definition of $U_{Q\vee\neg Q}$,

$$\begin{aligned}(\exists t \in \mathsf{Tree}(1))(\forall z \in \mathsf{Branch}(1,t))[\mathsf{lf\_br}(1,t,z) = 0 \vee{}&\\ (\mathsf{lf\_br}(1,t,z) = 1 \,\&\, (Q \vee \neg Q))]&\end{aligned} \tag{5}$$

Uniformity in the model gives the existence of $t_0 \in \mathsf{Tree}(1)$ such that

$$(\forall z \in \mathsf{Branch}(1,t_0))[\mathsf{lf\_br}(1,t_0,z) = 0 \vee (\mathsf{lf\_br}(1,t_0,z) = 1 \,\&\, (Q \vee \neg Q))] \tag{6}$$

for all propositions Q. Let br be an arbitrary element in $\mathsf{Branch}(1,t_0)$. If $\mathsf{lf\_br}(1,t,br) = 1$ then, by (6), $Q \vee \neg Q$ for all Q, which is false in the ω-model. So, $\mathsf{lf\_br}(1,t,br) = 0$ for any br in $\mathsf{Branch}(1,t_0)$). Since $U_P(0)$ is true for all P, we then obtain

$$(\exists t \in \mathsf{Tree}(1))(\forall z \in \mathsf{Branch}(1,t))U_P(\mathsf{lf\_br}(1,t,z)$$

Hence, by (3), $1 \,\epsilon\, U''_P$ and we get $\neg\neg P$ for all propositions P, which is false.

Since this topology cannot have an axiom-set, we don't know a priori if the product topology of the space with itself is definable in type theory. This can be formulated as the problem whether an explicitly given monotone operator on subsets of $S = \{0,1\} \times \{0,1\}$ has a least fixed point in type theory. Given $U \subseteq S$ and $z \in \{0,1\}$ let $p_z(U)$ be the set of y such that $(z,y) \in U$ and $q_z(U)$ be the set of x such that $(x,z) \in U$.

Open Problem 4.5 *Let $F(U)$ be the set of (x,y) such that $y \lhd p_x(U)$ or $x \lhd q_y(U)$. This is a monotone operator on $\mathcal{P}(S)$ which is definable by a first-order formula, and which satisfies $U \subseteq F(U)$. Can we define in type theory the least fixed point of F containing U?*

The operator F is positive but not strictly positive. Such operators have been analyzed in [BFPS81].

It is interesting to note that it is possible to give a formulation of the Dedekind-MacNeille cover which is classically equivalent to the one above and for which an axiom-set can be found. In particular, the example above of a cover without an axiom-set has a classically equivalent formulation which has an axiom-set. In fact, let us define a new infinitary relation $\lhd'_{DM}$ over the set S by putting, for any $a \in S$ and $U \subseteq S$,

$$a \lhd'_{DM} U \equiv (\forall y \in S)\ (\neg(a \leq y) \rightarrow (\exists u \varepsilon U)\ \neg(u \leq y))$$

This definition is clearly classically equivalent to the definition of the cover $\lhd_{DM}$ above since $a \varepsilon \bigcap_{U \subseteq \downarrow y} \downarrow y$ if and only if $(\forall y \in S)\ (((\forall u \varepsilon U)\ u \leq y) \rightarrow (a \leq y))$.

Now, it is possible to prove in a straightforward way that $\lhd'_{DM}$ satisfies *reflexivity* and *transitivity* while to prove that it satisfies also $\leq$-*left* it is necessary to use *transitivity* of $\leq$. Finally in order to prove the validity of $\leq$-*right*, some more conditions must be required on the order relation. Indeed, according to the definition of $\lhd'_{DM}$, it is sufficient that if $\neg(u \leq y)$ and $\neg(v \leq y)$ then there exists z such that $z \leq u$, $z \leq v$ and $\neg(z \leq y)$. This condition is for instance satisfied if the pre-order relation $\leq$ is *total*, that is, for all $x, y \in S$, $(x \leq y) \vee (y \leq x)$ holds, as it was for the boolean algebra of two elements that we used above to build a formal topology which does not have an axiom-set.

After proving that $\lhd'_{DM}$ is a cover relation, the quickest way to see that it has an axiom-set is to observe that by putting, for any $a, y \in S$,

$$y \Vdash a \equiv \neg(a \leq y)$$

the very definition of $\lhd'_{DM}$ shows that it is representable and hence, by the results in section 4.4, it is inductively generable by using the following axiom-set:

$$\begin{aligned} I(a) &\equiv \{g \in \{y \in S | \neg(a \leq y)\} \rightarrow S |\ (\forall x \in \{y \in S | \neg(a \leq y\})\ \neg(g(x) \leq x)\} \\ C(a, f) &\equiv \mathsf{Im}[f] \end{aligned}$$

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