Constructive Mathematics and Higher Topos Theory

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Starting point of this work

Discussions about models of choice sequences

M.P. Fourman, *Continuous truth I, non-constructive objects.*
Logic Colloquium: Florence, 1982, pp. 161-180.

A.S. Troelstra, D. van Dalen, *Constructivism in Mathematics, Vol. 2*
Elsevier, 1984

Chuangjie Xu and M.H. Escardo. *A constructive model of uniform continuity.*
TLCA 2013, LNCS 7941, pp. 236-249.

Brouwer and Topology

1914 Hausdorff

Brouwer had already done his work in topology

Brouwer never adopted the standard approach but introduced his own notion

Notion of spread, not based on set theory

content versus *structuralism*

Conceptualisation: Kleene-Vesley 1965, Kreisel-Troelstra 1970

Sheaf model: Fourman, Grayson, Moerdijk

Cantor space

$\alpha, \beta, \ldots$ points of Cantor space $\mathbf{2}^{\mathbb{N}}$

"concrete side"

Observable: elements $a, b, c, \ldots$ of the Boolean algebra C generated by countably many symbols $p_0, p_1, \ldots$

"formal side"

$\alpha \Vdash a$ states that the point α is in the clopen defined by a

The elements of C are the "gauge" or "tests"

Cantor space

There is a map $m : \mathsf{C} \to \mathbb{N}$ (modulus) where $m(a)$ is the least k such that a belongs to the subalgebra generated by $p_0, \ldots, p_{k-1}$

We define $\alpha =_k \beta$ to mean $\alpha(n) = \beta(n)$ for all $n < k$

We have $\forall (\alpha\ \beta : \mathbf{2}^{\mathbb{N}}) \qquad \alpha =_{m(b)} \beta \ \to \ (\alpha \Vdash b) = (\beta \Vdash b)$

Formal statement of uniform continuity for the test functions $\Vdash b\ : \mathbf{2}^{\mathbb{N}} \to \mathbf{2}$

Stone spaces

In formal/point-free topology, Cantor space is defined to be C

The category of Stone spaces is defined to be the *opposite* of the category of Boolean algebras

B covered by $B[1/e_1],\dots,B[1/e_k]$ if $e_1,\dots,e_k$ partitition of unity of B

This defines a *site*

We work in the sheaf model over this site

Sheaves over the site of Stone spaces

What is $\mathbf{2}$ in this model?

$\mathbf{2}(X)$ is the set of elements of the space/Boolean algebra X

What is $\mathbf{2}^{\mathbb{N}}$ in this model?

$(\mathbf{2}^{\mathbb{N}})(X)$ should be the set of sequences of elements of X

So it is exactly the set $X \to \mathsf{C}$

So $\mathbf{2}^{\mathbb{N}}$ is *representable* by C

Sheaves over the site of Stone spaces

This implies that we have, for *any* presheaf F

$F^{\mathsf{C}}(X) = F(\mathsf{C} \times X)$

In particular

$\mathbf{2}^{\mathsf{C}}(X) = \mathbf{2}(\mathsf{C} \times X)$

An element of $\mathbf{2}(\mathsf{C} \times X)$ is of the form $\Sigma e_i c_i$ where e_i partition of unity of X

We can define the global modulus of uniform continuity functional $m : \mathbf{2}^{\mathsf{C}} \to \mathbb{N}$

$$\Sigma e_i c_i \longmapsto \Sigma e_i m(c_i)$$

Sheaves over the site of Stone spaces

We have a uniform continuity functional in the model

$$\forall(F : \mathsf{C} \to \mathbf{2})\forall(\alpha\ \beta : \mathsf{C})\ (\alpha =_{m(F)} \beta) \to F(\alpha) = F(\beta)$$

Sheaves over the site of Stone spaces

Note that this model can be seen as a Boolean version of the Zariski topos

Zariski topos: all commutative rings

R is covered by $R[1/e_1],\dots,R[1/e_k]$ if $1 = \langle e_1,\dots,e_k\rangle$

(So uniform continuity holds for the Zariski topos)

Note that Cantor space is a non finitely presented Boolean algebra

Sheaves over the site of Stone spaces

In this model, C has enough points

$$\forall (\alpha : \mathsf{C}) \quad \alpha \Vdash a \ \rightarrow \ \alpha \Vdash U$$

is equivalent to: a belongs to the ideal generated by U

Connections with other presentations

We can work with the full subcategory of finite powers of Cantor spaces

At any stage, we have introduced a finite number of choice sequences

We can always introduce an “independent” one

All these spaces are actually isomorphic

We can work with endomorphisms on Cantor spaces

Dependent type theory

Presheaf models

D. Scott *Relating models of λ-calculus*, 1980

M. Hofmann *Syntax and semantics of dependent type theory*, 1997

We build a cwf structure

A context is a presheaf $\Gamma(X)$ with restriction maps

A type over Γ is a presheaf over the category of elements of Γ

Dependent type theory

Natural definitions of $\Pi(x:A)B,\ \Sigma(x:A)B$ inductive types

Natural definitions of *universes*: $U_n(X)$ is the set of all n-small presheaves on the category of elements of (the presheaf represented by) X

Dependent type theory

Over the category of Stone spaces we have a particular family of types $R(c)\ (c : C)$

$C(X)$ is the set of covering of X (partitition of unity)

$R(X, c)$ is the set of all sieves defined by the covering c

A presheaf A is a sheaf when all maps $m_c^A : A \to A^{R(c)}$ are *isomorphisms*

Dependent type theory

We can define this notion internally $S : U_n \to \Omega$

We have $(\forall (x : A) S(B)) \to S(\Pi (x : A) B)$

We have $S(A) \to (\forall (x : A) S(B)) \to S(\Sigma (x : A) B)$

However if $V_n \subseteq U_n$ is the subobject of U_n defined by S we don't have $S(V_n)$

The universe of sheaves is not a sheaf!

A related question is discussed in EGA 1, 3.3.1

Dependent type theory

Here is one way to solve this problem in a constructive setting

Dependent type theory

In previous works, we have shown how to build models of univalence (and higher inductive types) as submodels of presheaf category over a base category $\mathcal{C}$

This simplifies in special cases the presentation of model structures as considered by Cisinski (the extra assumptions are: connections and base category closed by finite products) that are proved to be *complete* in this case

Dependent type theory

Let $\mathcal{S}$ be the category of Stone spaces

We work with presheaves over $\mathcal{C} \times \mathcal{S}$

We have a model of type theory with univalence

Model of univalent type theory

One description uses the internal language of presheaf models

cf. Agda formalisation of I. Orton and A. Pitts

We define a dependent type $F(A)$ (A is fibrant) which has appropriate closure properties and we can define internally a model of univalent type theory

Dependent type theory

In this model we have a family $R(c)$ $(c : C)$ and we can define (internally)

$$S(A) \;:=\; \Pi(c : C)\ \mathsf{isEquiv}\ m_c^A$$

Then we can prove as before

$(\forall (x : A) S(B)) \rightarrow S(\Pi(x : A) B)$

$S(A) \rightarrow (\forall (x : A) S(B)) \rightarrow S(\Sigma(x : A) B)$

But now we have $S(\Sigma(X : U_n) S(X))$!

Dependent type theory

$A \mapsto A^{R(c)}$ is an example of a *left exact modality*

Idempotent monads

The crucial role of these operators was stressed early on by M Shulman (2012)

Internal model

Given $S(A)$ we define an *internal model* of type theory

$[\![A]\!] = [A].1$

$x_1 : A_1, \dots, x_n : A_n \vdash M : A$ is interpreted by

$x_1 : [\![A_1]\!], \dots, x_n : [\![A_n]\!] \vdash [M] : [\![A]\!]$

Internal model

$$[x] = x$$

$$[\lambda(x : A)M] = \lambda(x : [[A]])[M]$$

$$[M\ \ N] = [M]\ [N]$$

$$[U_k] = (\Sigma(X : U_k)S(X), pf)$$

$$[\Pi(x : A)B] = (\Pi(x : [[A]])[[B]], pf)$$

$$[\Sigma(x : A)B] = (\Sigma(x : [[A]])[[B]], pf)$$

See Chapter 4 of the thesis of K. Quirin

Interest for constructive mathematics

If A is a sheaf of abelian groups over a space X, we want to define $H^1(X, A)$ and $H^2(X, A)$

One can define $H^1(X, A)$ as a groupoid: the groupoid of A-torsors

$H^2(X, A)$ as a 2-groupoid: the type of gerbes of band A

For this we need a way to talk about "sheaf" at the groupoid level and "sheaf" at the 2-groupoid level in a constructive setting

Interest for constructive mathematics

Grothendieck, letter to L. Breen 1975

The construction of the cohomology of a topos in term of integration of stacks makes no appeal at all to complexes of abelian sheaves and still less to the technique of injective resolutions

Deligne, *Le symbole modéré*, 1991

This language of torsors and gerbes is essentially equivalent to the one of cocycles of Čech. It is more convenient and more intrinsic if we accept to talk about objects defined up to isomorphisms, and category defined up to an equivalence unique up to unique isomorphisms, and glueing of stacks

Inductive types

These are defined using higher inductive types, e.g. N has constructors

$$0 : N \qquad\qquad S : N \to N$$

and

$$glue_c : N^{R(c)} \to N \qquad\qquad g_c : \Pi(n : N)\ n = glue_c\ \overline{n}$$

where $\overline{n} = \lambda(x : R(c))n$