On a model of choice sequences

Introduction

For application in algebraic geometry, Grothendieck introduced a generalization of the notion of sheaf. The notion of sheaf, introduced previously by Leray and Cartan, was defined over a *topological space* and Grothendieck generalized the notion of topological space to his notion of topos, defined over a *site*. It is quite remarkable that the analysis of the notion of choice sequence by Kreisel and Troesltra [4], involves implicitely a sheaf model also defined over a (non trivial) site [2]. Previously, Beth had (independently of Leray and Cartan) introduced the notion of Beth models, which is a particular case of a sheaf over a topological space, for analysing models of intuitionistic first-order logic.

The goal of this note is to present a variation of the Kreisel-Troelstra model for choice sequences over *Cantor* space and not over Baire space. It was motivated by the work [3], and the point is to give another presentation which exhibits an analogy with the notion of the big Zariski topos in algebraic topology [1].

1 Description of the site and key property

We take for our base category the opposite of the category of *Boolean algebra*. A covering of a Boolean algebra B is given by a partition of unity $e_1, \dots, e_n$ of B and the corresponding maps $B \to B[1/e_i]$, where $B[1/e_i]$ is the localisation of B at e_i, which can also be described as the quotient $B/(1 - e_i)$.

We denote by C the Boolean algebra corresponding to Cantor space by Stone duality. It can be described as the Boolean algebra freely generated by countably many element. If follows from this characterisation that we have a natural bijection $(C \to B) \simeq B^{\mathbb{N}}$.

Theorem 1.1 *In the sheaf model over this site,* 2 *is defined as the sheaf* $2(B) = B$ *and* $2^{\mathbb{N}}$ *is represented by* C*.*

Proof. The first claim has a direct proof. The second claim then follows since $(2^{\mathbb{N}})(B) = B^{\mathbb{N}}$ is naturally in bijection with the set of maps $C \to B$. □

2 Uniform continuity holds in the sheaf model

Given $\varphi : 2^{\mathbb{N}} \times \mathbb{N} \to \Omega$, we explain why the following implication is valid

$$(\forall(\alpha : 2^{\mathbb{N}})\exists(n : \mathbb{N})\varphi(\alpha, n)) \to \exists(M : \mathbb{N})\forall(\alpha : 2^{\mathbb{N}})\exists(n \leqslant M)\varphi(\alpha, n)$$

Let us assume that the hypothesis $\forall(\alpha : 2^{\mathbb{N}})\exists(n : \mathbb{N})\varphi(\alpha, n)$ holds at stage B. This means that for *any* $f : B \to B'$ and $\alpha : C \to B'$ the formula $\exists(n : \mathbb{N})\varphi(\alpha, n)$ holds at stage B'. In turn, this means that we have a partition of unity e_i in B' and a corresponding family of numbers n_i such that $\varphi(\alpha, n_i)$ holds for $C \to B' \to B'[1/e_i]$. To validate the implication, it is thus enough to show that we can find a bound for the n_i independent of the choice of $B',\ f,\ \alpha$.

The key point is that to give $B', f : B \to B'$ and $\alpha : C \to B'$ is equivalent to give B' and a map $B + C \to B'$, where $B + C$ is the sum of B and C. Thus to compute an upper bound M we only have to look at the *particular* partition of unity obtained for the universal choice $B' = B + C$ and the maps $B \to B + C$ and $C \to B + C$. This last map can be thought of as the generic choice sequence at stage B.

3 Connection with Zariski site

The site that we have been using can be seen as a "Boolean" version of the (big) Zariski site [1], which consists of the category of *rings* where a covering of a ring R is given by a comaximal family of elements $1 = (x_1, \ldots, x_n)$ and the corresponding set of maps $R \to R[1/x_i]$.

4 How this model is usually presented

The only objects in the base category relevant for this model are finite products of localisations of Cantor space. All these objects are isomorphic to Cantor space. Thus, one can work with the equivalent category of Cantor space and endomorphisms. This is what is done in [4, 2] for Baire space and [3] for Cantor space. An advantage of our formulation is that Theorem 1.1 has a transparent proof.

References

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