Variation on Cubical sets

August 31, 2014

Introduction

In the model presented in [1, 4] a type is interpreted by a nominal set A equipped with two "face" operations: if $u : A$ and i is a symbol we can form $u(i = 0) : A(i = 0)$ and $u(i = 1) : A(i = 1)$ elements independent of i. The unit interval is represented by the nominal set $\mathbf{I}$, whose elements are $0, 1$ and the symbols. The set $\mathbf{I}$ however does not satisfy the Kan filling condition.

In this model, if A represents a type, the path space of A is represented by the affine exponential $\mathbf{I} \rightarrow *A$, which is adjoint to the separated product $B * I$ with $(b, x) \in B * I$ if x is independent of b.

This means in particular that if we have $u : I \rightarrow *A$ then we cannot in general apply u to an arbitrary symbol x.

For certain operations, e.g. the realization of function extensionality, and of the elimination rule for the circle, it seems that we get more natural realizers if we could represent the path space as a real exponential $\mathbf{I} \rightarrow A$.

One way to achieve this is to allow for operations on cubes not only swapping but also arbitrary substitutions.

In term of cubical sets [1], this amounts to work the category of finite sets with maps $I \rightarrow J + 2$ (i.e. the Kleisli category for the monad $I + 2$). This category appears on pages 47–48 in Pursuing Stacks [3] as "in a sense, the smallest test category".

The path space $\mathbf{I} \rightarrow A$ and if $t : A$ and i is a symbol we can form $\langle i \rangle t$ in $\mathbf{I} \rightarrow A$ which behaves now as lambda abstraction.

In this note we will adopt the following notations. We will use the letters $i, j, k, \ldots$ for symbols/colours/names and $p, q, \ldots$ for elements of $\mathbf{I}$ (that can be symbols or $0, 1$) and $b, c \ldots$ for $0, 1$. We write simply ip for the substitution $i = p$ so that $u(ip) : A(ip)$ if $u : A$. We present a generalization of the Kan filling operation, which provides a simple interpretation of identity type and dependent product. We also provide a definition of the Kan filling operation for the universe and give an interpretation of univalence. Computation rules for identity holds in a definitional way, and this is actually crucial in this model for the interpretation of univalence.

1 Kan condition

We define a J-*tube* for a type A and finite set of symbols J to be a family of elements $u_{ib} : A(ib)$ for i in$J$ with the compatibility condition $u_{ib}(jc) = u_{jc}(ib)$.

Given a J-tube $\vec{u}$ we define $\vec{u}(ip)$ (if i is in J then p has to be a symbol).

There are 3 main cases.

1. If i is not in J we write $v_{jb} = u_{jb}(ip(jb))$ and define $\vec{u}(ip)$ to be $\vec{v}$
2. If i is in J and $p = k$ is a symbol not in J we define the following $J - i, k$-tube by taking $v_{jb} = u_{jb}(ik)$ if $j \neq i$ and $v_{kb} = u_{kb}(ib)$
3. If i is in J and $p = k$ is a symbol in J we define the following $J - i$-tube by taking $v_{jb} = u_{jb}(ik)$ if $j \neq i, k$ and $v_{kb} = u_{kb}(ib)$

We say that a type A *satisfies the Kan property* if we have the following operation. Given a J-tube $\vec{u}$ a symbol i not in J and a in $A(ip)$ compatible with $\vec{u}(ip)$, i.e. such that $a(jb) = u_{jb}(ip(jb))$ we can compute by a given operation

$$a' = \mathsf{comp}^i_{A,\vec{u}}(p, q, a)$$

an element a' in $A(iq)$ also compatible with $\vec{u}(iq)$, so such that $a'(jb) = u_{jb}(iq(jb))$.

In this operation, we consider i to be bound, so that we have

$$\mathsf{comp}^i_{A,\vec{u}}(p, q, a) = \mathsf{comp}^k_{A(ik),\vec{u}}(p, q, a)$$

if k is fresh, and we can assume w.l.o.g. that p, q, a are independent of i. This operation also has to satisfy

$$\mathsf{comp}^i_{A,\vec{u}}(p, q, a) = a$$

if $p = q$ (and if $A, \vec{u}$ are independent of i).

The uniformity condition is that we have

$$a'(kr) = \mathsf{comp}^i_{A(kr),\vec{u}(kr)}(p(kr), q(kr), a(kr))$$

where k is a symbol $\neq i$, but which can be in J if r is a symbol.

One key point is that only i is bound in this composition operation; the symbols in J are free. In particular if j, k are in J then $a'(jk)$ will itself be a composition for the tube $\vec{u}(jk)$. This seems crucial in order to get a simple computation for the composition of a dependent product.

Intuitively the symbols that appear freely in $\mathsf{comp}^i_{A,\vec{u}}(p, q, a)$ are the ones that appear freely in $A, \vec{u}, p, q, a$ minus i. Intuitionistically if j is independent of in $A, \vec{u}, p, q, a$ or $j = i$ then j is independent of $\mathsf{comp}^i_{A,\vec{u}}(p, q, a)$.

A first application of this operation is to define a generalization of the usual Kan filling operation. If we have a J-tube $\vec{u}$ and an element $a : A(ip)$ compatible with $\vec{u}(ip)$ then we can find $\tilde{a} : A$ compatible with $\vec{u}$ and such that $\tilde{a}(ip) = a$. For this, we take $\tilde{a} = a'(ki)$ for a fresh k with

$$a' = \mathsf{comp}^i_{A,\vec{u}}(p, k, a)$$

Notice that we then have, by uniformity condition

$$\tilde{a}(iq) = a'(ki)(iq) = a'(kq) = \mathsf{comp}^i_{A,\vec{u}}(p, q, a)$$

We write $\tilde{a} = \mathsf{fill}^i_{A,\vec{u}}(p, a)$

2 Dependent product

Given a J-tube $\vec{\mu}$ for $\Pi\ A\ F$ and λ in $\Pi\ A(ip)\ F(ip)$ we want to define

$$\lambda' = \mathsf{comp}^i_{\Pi\ A\ F,\vec{\mu}}(p, q, \lambda)$$

We know, and this is a key point, a priori what $\lambda'(kr)$ should be (defined by a composition or directly as μ_{kb} if k is in J and $r = b$). So we only have to define what is

$$\lambda'\ a$$

for a in $A(iq)$. We first use the operation of Kan completion for A getting an element $\tilde{a} : A$ such that $\tilde{a}(iq) = a$. We then define $v_{jb} = \mu_{jb}\ \tilde{a}(jb)$ and

$$\lambda'\ a = \mathsf{comp}^i_{F\ \tilde{a},\vec{v}}(p, q, \lambda\ \tilde{a}(ip))$$

3 Identity type

The Kan operation for identity type is similar to the one in [1].

4 Function extensionality

5 Propositional truncation

6 Equivalence

The following property of equivalence $\sigma : A \rightarrow B$ will be crucial.

Let v in B such that there exists a L-tube $\vec{a}$ in A such that $v_{lb} = \sigma_{lb} a_{lb}$. Then there exists a filling a of $\vec{a}$ (i.e. an element a in A such that $a(lb) = a_{lb}$) and $\tilde{b}$ such that $\tilde{b}(i1) = b$ and $\tilde{b}(i0) = \sigma a$.

7 Glueing

We need now to define how to transform an equivalence to an equality of types and what is the composition for the universe. Both definitions use the same computation, which can be described as a "glueing" operation. We assume given a type A and for each l in L an equivalence $\sigma_{lb} : T_{lb} \rightarrow A(lb)$. We explain how to define a new type $A' = (\vec{T}, A)$ such that $A'(lb) = T_{lb}$. Intuitively we replace the (lb)-face of A by T_{lb} using the equivalence σ_{lb}.

An element of A' is of the form $(\vec{t}, a)$ with a in A and t_{lb} in T_{lb} such that $a(lb) = \sigma_{lb} t_{lb}$. If all σ_{lb} are identities we have $a(lb) = t_{lb}$ and we identify a with $(\vec{t}, a)$ and A' with A.

8 Transforming an equivalence to an equality

We assume given two types A, B and a colour i and an equivalence $\sigma : A \rightarrow B(i0)$. We explain then how to define a type E such that $E(i0) = A$ and $E(i1) = B$. Intuitively we replace the $(i0)$-face of B by A using the equivalence σ.

An element of $E = (A,_i B)$ is a pair $(a,_i v)$ with $a : A,\ v : B$ and $\sigma a = v(i0)$. We define $E(jq)$ by case on j. We have $E(i0) = A,\ E(i1) = B(i1)$ and $E(ik) = (A',_k B')$ with $B' = B(ik)$ and $A' = A(k0)$ and $\sigma' = \sigma(k0)$. If $j \neq i$ then $E(jq) = (A(jq(i0)),_i B(jq))$. Similarly we have $(a,_i v)(i0) = a,\ (a,_i v)(i1) = v(i1)$ and $(a,_i v)(ik) = (a(k0),_k v(ik))$. If $j \neq i$ then $(a,_i v)(jq) = (a(jq(i0)),_i v(jq))$.

We explain now how to define $\mathsf{comp}^k_{E,\vec{u}}(p, q, (a,_i v)) : E(kq)$ in the case $k \neq i$ and $\vec{u}$ is a J-tube with i in J. We have $a : A(kp(i0))$ and $v : B(kp)$ and $\sigma(kp(i0))a = v(i0)$. We also have

$$u_{jb} = (a_{jb},_i v_{jb})$$

for $j \neq i$ and

$$u_{i0} = a_{i0} : A \qquad\qquad u_{i1} = v_{i1} : B(i1)$$

We define

$$a' = a_{i0}(kq(i0)) : A(kq(i0))$$

and

$$v' = \mathsf{comp}^k_{B,\vec{v}}(p, q, v) : B(kq)$$

where $v_{i0} = \sigma a_{i0} : B(i0)$.

If i is not in J we define

$$a_{i0} = \mathsf{fill}^k_{A,\vec{a}}(p(i0), a)$$

and

$$v_{i1} = \mathsf{fill}^k_{B(i1),\vec{v}(i1)}(p(i1), v(i1))$$

and add the pair $u_{i0} = a_{i0},\ u_{i1} = v_{i1}$ to the J-tube.

We explain how to compute $\mathsf{comp}^i_{E,\vec{u}}(p, q, e) : E(iq)$ $\vec{u}$ is a J-tube for E.

We look at the computation of $\mathsf{fill}^i_{E,\vec{u}}(k, e) : E$ with $e : E(ik)$ and k is in J. We have $e = (a,_k v)$ with $a : A(k0)$ and $v : B(ik)$ and $v(k0) = \sigma(k0)a$. We also have $u_{jb} : E(jb)$ so that we can write

$$u_{jb} = (a_{jb},_i v_{jb}) \qquad a_{jb} : A(jb) \qquad v_{jb} : B(jb) \qquad v_{jb}(i0) = \sigma(jb)a_{jb}$$

Since e is compatible with $\vec{u}(ik)$ we should have $e(jb) = u_{jb}(ik(jb))$ for all j in J and so if $k \neq j$

$$a(jb) = a_{jb}(k0) \qquad v(jb) = v_{jb}(ik)$$

and for $k = j$ we have $e(kb) = u_{kb}(ib)$, so that $a = a_{k0}$ and $v(k1) = v_{k1}(i1)$. We can then form

$$\tilde{v} = \mathsf{fill}^i_{B,\vec{v}}(k, v)$$

The problem with this element is that $\tilde{v}(i0)$ does not need to be in the image of σ. However all elements of the boundary

$$\tilde{v}(i0)(jb) = v_{jb}(i0) = \sigma(jb)a_{jb}$$

are in the image of $\sigma(jb)$. Since σ is an equivalence the face $\tilde{v}(i0)$ is homotopic modulo its boundary to an element in the image of σ and we perform a glueing replacing the face $\tilde{v}(i0)$ by this image.

The last case is for the computation of of $\mathsf{fill}^i_{E,\vec{u}}(k, e) : E$ with $e : E(ik)$ and k is not in J. In this case we first add k to J. We have to compute a_{kb} and v_{kb}.

9 Universe

If we have a J-tube $\vec{T}$ in U and A compatible with $\vec{T}(ip)$, that is $A(jb) = T_{jb}(ip(jb))$ for all j in J, we define

$$A' = \mathsf{comp}^i_{U,\vec{T}}(p, q, A)$$

to be the type of elements of the form $(\vec{u}, a)$ with $a : A$ such that

$$a(jb) = \mathsf{comp}^i_{T_{jb}}(q(jb), p(jb), u_{jb})$$

If $\vec{T}$ is independent of i then we should have $a(jb) = u_{jb}$. In this case, we identify a with $(\vec{u}, a)$.

The composition operations for A' are defined in a similar way to the previous ones for equality $(A,_i B)$. The only difference is that for

$$A' = \mathsf{comp}^i_{U,\vec{T}}(p, q, A)$$

we replace that (jb)-face of A, which is $A(jb) = T_{jb}(ip(jb))$, by the face $T_{jb}(iq(jb))$. We also have a map

$$\sigma_{jb} : T_{jb}(iq(jb)) \to T_{jb}(ip(jb)) \qquad v \longmapsto \mathsf{comp}^i_{T(jb)}(q(jb), p(jb), v)$$

and what we need is that this map is an equivalence.

Lemma 9.1 *For any type A and colour i the map $A(ip) \to A(iq)$, $u \longmapsto \mathsf{comp}^i_A(p, q, u)$ is an equivalence.*

References

[1] M.Bezem, Th. Coquand and S. Huber. A model of type theory in cubical sets. Preprint, 2013.

[2] M.J. Gabbay and M. Hofmann. Nominal renaming sets. Logic for Programming, Artificial Intelligence, and Reasoning Lecture Notes in Computer Science Volume 5330, 2008, pp 158-173.

[3] A. Grothendieck. Pursuing stacks. Manuscript, 1983.

[4] A. M. Pitts. An Equivalent Presentation of the Bezem-Coquand-Huber Category of Cubical Sets. Manuscript, 17 September 2013.