A model structure on some presheaf categories

Introduction

The goal of this note is to present a general class of model structures on some presheaf categories. This is due to Christian Sattler relying on the «glueing» operation of the cubical set model.

1 Base category

We write $I, J, K, \ldots$ the objects of a given small category $\mathcal{C}$ with finite product. We assume given a special object $\mathbb{I}$ with two maps $0, 1 : T \to \mathbb{I}$ where T is the terminal object. This defines a *cylinder functor*: if we write $J^+ = J \times \mathbb{I}$, we have natural transformations $p : J^+ \to J$, and $e_0, e_1 : J \to J^+$ such that $pe_0 = pe_1 = 1_J$. [1]

A (generalized) *cubical set* is a presheaf on $\mathcal{C}$.

If Γ is a cubical set, and $f : J \to I$ we write

$$u \longmapsto uf \qquad \Gamma(I) \to \Gamma(J)$$

the corresponding restriction map, and we have $u1_I = u$ and $(uf)g = u(fg)$ if $f : J \to I$ and $g : K \to J$.

We identify an object I and the cubical set it represents.

We have the cubical set Ω, where $\Omega(I)$ is the set of *sieves* on I.

We assume given a sublattice $\mathbb{F}$ of Ω, with the only condition that we have a natural map $c_0 : I^+ \to \mathbb{F}$ (resp. $c_1 : I^+ \to \mathbb{F}$) which classifies e_0 (resp. e_1). If we want the model to be effective, we also require that it is decidable whether a sieve in $\mathbb{F}(I)$ is equal to the total sieve or not.

If Γ is a cubical set, any map $\psi : \Gamma \to \Omega$ defines a subobject Γ, ψ of Γ with a canonical mono $\iota_\psi : \Gamma, \psi \to \Gamma$.

2 Model structure

We define first the notion of (generalized) open boxes. Given a cubical set A and $\psi : A \to \mathbb{F}$ we define $i_0(A, \psi) : b_0(A, \psi) \to A^+$ to be the mono classified by $\psi p \vee c_0$ (and similarly $i_1(A, \psi) : b_1(A, \psi) \to A^+$).

Definition 2.1 We define 4 classes of maps.

1. A map is a *cofibration* iff it is classified by $\mathbb{F}$
2. a map is a *trivial fibration* iff it has the right lifting property w.r.t. any cofibration
3. a map is a *fibration* iff it has the right lifting property w.r.t. any maps $i_0(A, \psi)$ and $i_1(A, \psi)$ for any A and $\psi : A \to \mathbb{F}$
4. a map is a *trivial cofibration* iff it has the left lifting property w.r.t. any fibration

[1] We also need to assume *connection* operations that are natural transformations $m, j : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ such that $j(x, 0) = j(0, x) = x$ and $m(x, 1) = m(1, x) = x$, and, for the glueing operation, we also need to assume an operation $\forall : P(\mathbb{F}) \to \mathbb{F}$ with $P(A)(I) = A(I^+)$ and $\psi \leqslant \forall \delta$ iff $\psi p \leqslant \delta$. Because of the need of connections, this class of model does not contain Grothendieck's «smallest» test category, since there any map $\mathbb{I}^n \to \mathbb{I}$ has to be constant or a projection.

Notice that both fibration properties are purely existential. They also are defined by right lifting properties, and a priori, may not be stable by addition of new Grothendieck universes. As we will see, these properties are actually equivalent to the existence of corresponding structures of a fixed size, which also imply that the two notions of fibrations are «absolute», i.e. stays the same even if we add new universes,

If $f : A \to B$ we define C and $i_f : A \to C$ and $p_f : C \to B$ by taking $C(I)$ to be the set of tuples (b, ψ, u) with b in $B(I)$ and ψ in $\mathbb{F}(I)$ and $u : I, \psi \to A$ such that $fu = b\iota_\psi$. We define $i_f\ a = (f\ a, 1, a)$ and $p_f\ (b, \psi, u) = b$.

Proposition 2.2 *We have $f = p_f i_f$ and i_f is a cofibration (it is classified by $g : C \to \mathbb{F}$ defined by $g(b, \psi, u) = \psi$) and p_f is a trivial fibration*

We define f to be a *weak equivalence* iff the map $i_f : A \to C$ is a *trivial* cofibration.

Theorem 2.3 *(Christian Sattler) The three classes of maps* cofibration, fibration, weak equivalence *define a model structure on cubical sets.*

His argument is an application of the model of type theory with universe and univalence we present below.

It also follows from his argument that the notion of weak equivalence is absolute and hence that the notion of trivial cofibration is absolute as well (this is not obvious a priori since it is defined by having a left lifting property w.r.t. a class a maps which changes when one adds universes).

3 Dependent type

If Γ is a cubical set, we can consider its *category of elements* $\int \Gamma$: an object is of the form (I, ρ) with ρ in $\Gamma(I)$ and a map $f : (J, \nu) \to (I, \rho)$ is a map $f : J \to I$ such that $\nu = \rho f$ in $\Gamma(J)$.

A *dependent type* on Γ, notation $\Gamma \vdash A$, is a presheaf on the category of elements of Γ.

If $\sigma : \Delta \to \Gamma$ then σ determines a functor $\int \Delta \to \int \Gamma$ sending (I, ν) to $(I, \sigma\nu)$ and $\Gamma \vdash A$ determines by composition a dependent type $\Delta \vdash A\sigma$.

If we have $\Gamma \vdash A$ we can define a new cubical set $\Gamma.A$ by taking $(\Gamma.A)(I)$ to be the set of elements ρ, u with ρ in $\Gamma(I)$ and u in $A(I, \rho)$ and $(\rho, u)f = (\rho f, uf)$. We have a canonical map $p_A : \Gamma.A \to \Gamma$ defined by $p_A(\rho, u) = \rho$.

We define $\Gamma \vdash u : A$ to mean that $u : \Gamma \to \Gamma.A$ is a section of p_A.

Though the presheaf category on $\int \Gamma$ and the slice category over Γ are equivalent, it is important to distinguish them to be able to state results with strict equality (which is crucial to get a model of type theory in a simple way without coherence issues).

4 Contractible structure

We first define what is a *contractible* structure on a given cubical set A.

If $\psi : I \to \mathbb{F}$ we recall that we write $\iota_\psi : I, \psi \to I$ the associated mono it classifies. If $f : J \to I$ we can pull-back this map along ι_ψ, getting a map $\psi^* f : J, \psi f \to I, \psi$.

A contractible strucure is given by an *extension* operation $\mathsf{ext}(I, \psi, u) : I \to A$ where ψ is an element of $\mathbb{F}(I)$ and $u : I, \psi \to A$, which satisfies $\mathsf{ext}(I, \psi, u)\iota = u$ *and* the uniformity condition

$$\mathsf{ext}(I, \psi, u)f = \mathsf{ext}(J, \psi f, u\psi^* f)$$

To have such an extension operation is actually *equivalent* to the fact that given *any* cubical set B and any map $\psi : B \to \mathbb{F}$, a map $u : B, \psi \to A$ can be extended to a map $\overline{u} : B \to A$ (such that $\overline{u}\iota_\psi = u$). More precisely, to give an extension operation is the same as giving a retraction of the canonical map $A \to \tilde{A}$ where $\tilde{A}(I)$ is the set of elements I, ψ, u with ψ in $\mathbb{F}(I)$ and $u : I, \psi \to A$.

5 Acyclic Kan structure

An *acyclic Kan* structure on a dependent type $\Gamma \vdash A$ is an extension operator $I \vdash \mathsf{ext}(\rho, \psi, u) : A\rho$ such that $\mathsf{ext}(\rho, \psi, u)f = \mathsf{ext}(\rho f, \psi f, u\psi^* f)$ and $\mathsf{ext}(\rho, \psi, u)\iota_\psi = u$ given ρ in $\Gamma(I)$ and $\psi : I \to \mathbb{F}$ and $I, \psi \vdash u : A\rho\iota_\psi$ and $f : J \to I$.

To have such a structure is logically equivalent to the fact that p_A has the right lifting property w.r.t. *any* mono classified by $\mathbb{F}$.

6 Kan structure

If $\sigma : B \to A$ and $\psi : A \to \mathbb{F}$ we have maps

$$b_0(\sigma) : b_0(B, \psi\sigma) \to b_0(A, \psi) \qquad b_1(\sigma) : b_1(B, \psi\sigma) \to b_1(A, \psi)$$

We define what is a *fibration structure* on a dependent type $\Gamma \vdash A$. It is given by two «filling» operations, giving the *uniform* lifting properties w.r.t. the inclusions $i_0(J, \psi)$ and $i_1(J, \psi)$ for $\psi : J \to \mathbb{F}$. Thus we have $J^+ \vdash \mathsf{fill}_0(J, \psi, \rho, u) : A\rho$ given $\psi : J \to \mathbb{F}$ and $\rho : J^+ \to \Gamma$ and $b_0(J, \psi) \vdash u : A\rho i_0(J, \psi)$ satisfying $\mathsf{fill}_0(J, \psi, \rho, u)i_0(J, \psi) = u$ *together with* the uniformity condition: if $g : K \to J$ then

$$\mathsf{fill}_0(J, \psi, \rho, u)g = \mathsf{fill}_0(K, \psi g, \rho g^+, ub_0(g))$$

(and similarly changing 0 to 1).

Proposition 6.1 *To have a fibration structure on $\Gamma \vdash A$ is logically equivalent to the fact that the map $p_A : \Gamma.A \to \Gamma$ have the right lifting property w.r.t. any maps $i_0(\Delta, \psi)$ and $i_1(\Delta, \psi)$ for $\psi : \Delta \to \mathbb{F}$ and for* any *cubical set Δ (not necessarily representable).*

If c_A is a fibration structure on $\Gamma \vdash A$ and $\sigma : \Delta \to \Gamma$ then we define a fibration structure $c_A\sigma$ on $\Delta \vdash A\sigma$ by composition.

7 Equivalence structure

If $\Delta \vdash T$ and $\Delta \vdash A$ we write $\Delta \vdash w : T \to A$ to mean that w is a natural transformation between the two presheafs T and A on $\int \Delta$. We define the *homotopy fiber* $\Delta.A \vdash F_w$ by taking $F_w(I, \rho, u)$ for ρ in $\Delta(I)$ and u in $A(I, \rho)$ to be the set of elements (t, ω) where t is in $T(I, \rho)$ and ω an element of $A(I^+, \rho p)$ such that $\omega e_0 = w\ t$ and $\omega e_1 = u$. If $f : J \to I$ we define $(t, \omega)f = (tf, \omega f^+)$.

An *equivalence structure* c_w for the map w is then an acyclic Kan structure for $\Delta.A \vdash F_w$ (this expresses that each fiber of w is contractible).

8 Glueing operation

Theorem 8.1 *Given*

- *a dependent type $\Gamma \vdash A$*
- *a mono $\sigma : \Delta \to \Gamma$ classified by $\mathbb{F}$*
- *a dependent type $\Delta \vdash T$ and a natural transformation $\Delta \vdash w : T \to A\sigma$*

then we can find

1. *$\Gamma \vdash G$ such that $G\sigma = T$* (strict *equality)*
2. *a natural transformation $\Gamma \vdash e : G \to A$ such that $e\sigma = w$*
3. *if furthermore T has a Kan structure c_T and A has a Kan structure and w has an equivalence structure c_w, then we can find a Kan structure c_G for $\Gamma \vdash G$ such that $c_G\sigma = c_T$ and an equivalence structure c_e such that $c_e\sigma = c_w$.*

9 Universe

We suppose given a Grothendieck universe $\mathcal{U}$ and we assume that the base category is in $\mathcal{U}$.

We define a corresponding cubical set U by taking $U(I)$ to be the set of all pairs (A, c_A) where A is a $\mathcal{U}$-dependent type $I \vdash A$ and c_A is a fibration structure on $I \vdash A$. If $f : J \to I$ we define $(A, c_A)f$ to be Af, c_Af.

We can define $U \vdash El$ by taking $El(I, A, c_A)$ to be the set of all sections $I \vdash u : A$.

Theorem 9.1 *$U \vdash El$ has a canonical fibration structure c_E such that if $\Gamma \vdash A$ is a $\mathcal{U}$-dependent type with a fibration structure c_A, there exists a unique map $|A| : \Gamma \to U$ such that $El|A| = A$ and $c_E|A| = c_A$.*

10 The universe is fibrant

It follows from Theorem 8.1 that the cubical set U is fibrant, i.e. the map $U \to 1$ has a fibration structure. Christian Sattler has shown that this implies that a map is a trivial cofibration iff it is a cofibration which is a weak equivalence.

To provide a simple example, the fact that fibrations can be extended along trivial cofibrations can be refined in the following way.

Theorem 10.1 *If $\sigma : \Delta \to \Gamma$ is a trivial cofibration and $\Delta \vdash B$ is a $\mathcal{U}$-dependent type with a fibration structure c_B there exists a $\mathcal{U}$-dependent type $\Gamma \vdash A$ with a fibration structure c_A such that $A\sigma = B$ and $c_A\sigma = c_B$.*

Indeed, we have $B = El|B|$ and since U is fibrant and $\Delta \to \Gamma$ is a trivial cofibration there exists a map $|A| : \Gamma \to U$ such that $|A|\sigma = |B|$. We define then $A = El|A|$ and $c_A = c_E|A|$ and we have $A\sigma = El|A|\sigma = El|B| = B$ and $c_A\sigma = c_E|A|\sigma = c_E|B| = c_B$.

All the results of Sections 3 to 8 have been checked formally in NuPrl by Mark Bickford for a special case of the base category (where a morphism $J \to I$ is given by a set theoretic map from I to the free de Morgan algebra on J).