Infinite objects in constructive mathematics

Thierry Coquand

May 8 2007

Riesz space

Assume that the vector space E is an ordered space which is a lattice (automatically distributive) and that it contains a special element 1 which is a *strong unit*: for all $a \in E$, there exists n such that $a \leq n.1$

Example: $C([0,1])$

Then we can define $N(r)$ by $x \in N(p/q)$ iff $qx \leq p.1$ *and* $-qx \leq p.1$

No reason why $|x| = \inf \{r \mid x \in N(r)\}$ should computable (Dedekind real)

x is *normable* iff $|x|$ is a Dedekind real

Riesz space

We can define the space of integrals $I(E)$: points of $Fn(E)$ such that $u(1) = 1$

We can replace $u(a) < r$ by $0 < u(r.1 - a)$.

Generators $I(a)$ and relations $I(a) = 0$ if $a \leq 0$ and

$I(a) \wedge I(-a) = 0, \quad I(a + b) \leq I(a) \vee I(b), \qquad I(1) = 1$

$I(a) = \bigvee_{r>0} I(a - r.1)$

Spectrum of a Riesz space

We take the generators $D(a)$ and same relations

$D(a) = 0$ if $a \leq 0$

$D(a) \wedge D(-a) = 0, \quad D(a+b) \leq D(a) \vee D(b), \qquad D(1) = 1$

with the extra condition $D(a \vee b) = D(a) \vee D(b)$

We get a strongly normal lattice $Sp(E)$,

We add the relation $D(a) = \vee_{r>0} D(a-r)$

We get a compact space $X = Sp_r(E)$, subspace of $I(E)$. The space $I(E)$ can be thought of as the space of probability measure on X

Spectrum of a Riesz space

We have a complete description of $Sp(E)$; notice that $a \in (p, q)$ is definable as $D(a - p.1) \wedge D(q.1 - b) = D((a - p.1) \wedge (q.1 - a))$

We take the set P elements that are ≥ 0 in E

We define the new relations $a \leq' b$ iff there exists n such that $a \leq n.b$

P for this relation is a distributive lattice, and this is a concrete description of $Sp(E)$

Corollary: *We have $D(a) = 1$ in $Sp(E)$ iff there exists n such that $1 \leq na$.*

Real spectrum of a Riesz space

If $X = Sp_r(E)$ then X is compact regular

There is a dense norm preserving injection $E \rightarrow C(X)$ (Stone-Weirstrass)

This is a *representation theorem*

X is *overt* iff all elements of E are *normable* i.e. for all $x \in E$ we have that $|x|$ is a Dedekind real

Using this, one can obtain a proof of Gelfand representation theorem (for commutative algebra of operators) in Bishop style mathematics, simpler than Bishop-Bridges' proof

Space of valuations

Let L be a field (constructively $x = 0 \vee \exists y. xy = 1$)

We want a formal space whose points are the valuation rings of L

$[x \in A] \wedge [y \in A] \;\leq\; [x + y \in A] \wedge [xy \in A]$

$1 = [x \in A] \vee [1/x \in A]$ if $x \neq 0$

Interpret $[x \in A]$ symbolically: take the distributive lattice generated by these conditions

This defines a formal spectral space $V(L)$

Space of valuations

More generally if R is a subring of L we define the space $V_R(L)$ of valuation rings containing R by the theory

$$[x \in A] \wedge [y \in A] \ \leq \ [x+y \in A] \wedge [xy \in A]$$

$1 = [x \in A] \vee [1/x \in A]$ if $x \neq 0$

$1 = [x \in A]$ if $x \in R$

Space of valuations

Theorem: *We have* $[x_1 \in A] \wedge \cdots \wedge [x_n \in A] \leq [x \in A]$ *in the space* $V_R(L)$ *iff* x *is* integral *over* $R[x_1, \ldots, x_n]$

In term of points: the intersection of all valuation rings containing $x_1, \ldots, x_n$ is the set of elements integral over $R[x_1, \ldots, x_n]$

Space of valuations and Zariski spectrum

Let R be an integral domain and $L = \mathsf{Frac}(R)$

Theorem: *The lattice map* $D(x) \longmapsto [1/x \in A]$ *for* $x \neq 0$ *from the lattice* $Zar(R)$ *to the lattice* $V_R(L)$ *is* conservative

This is called the *center* map

Theorem: *If* R *is* arithmetical *the center map is an isomorphism*

R arithmetical iff the lattice of ideals is distributive iff for any x, y we can find u, v, w such that $xu = yw,\ y(1-u) = xv$

Riemann surface

Dedekind-Weber (1882); one early point-free description of a space

Let $L = \mathbb{Q}(x, y)$ with $y^2 = 1 - x^4$

We can consider the space X of valuation rings containing $\mathbb{Q}$

This is a spectral space, and it has a formal covering $X = U_0 \cup U_1$

$U_0 = [x \in A] \quad U_1 = [1/x \in A]$

Riemann surface

R_0 integral closure of $\mathbb{Q}[x]$ in L

R_1 integral closure of $\mathbb{Q}[1/x]$ in L

Theorem: *R_0 and R_1 are arithmetical ring*

Corollary: *$U_0 \equiv Zar(R_0)$ and $U_1 \equiv Zar(R_1)$*

Towards formal sheaf theory

Over a space $Zar(R)$ we have a sheaf of rings, called the *structure* sheaf $\mathcal{O}(D(a)) = R[1/a]$

If R integral domain we have $\mathcal{O}(D(a_1, \dots, a_n)) = R[1/a_1] \cap \cdots \cap R[1/a_n]$

The sheaf glueing property is what Henri calls the local-global principle

The structure $Zar(R), \mathcal{O}$ is called a (formal) *affine* scheme

Towards formal sheaf theory

Over the space X of valuations there is a natural sheaf

$\mathcal{F}([u_1 \in A] \wedge \cdots \wedge [u_n \in A])$ is the integral closure of $\mathbb{Q}[u_1, \ldots, u_n]$

The fiber at the point A is the ring A itself!

Over the open $U_0 = [x \in A]$ the sheaf $\mathcal{F}$ reduces to the structure sheaf over the ring R_0

Over the open $U_1 = [1/x \in A]$ the sheaf $\mathcal{F}$ reduces to the structure sheaf over the ring R_1.

We have a most natural example of a scheme: glueing of two affine scheme

Towards formal sheaf theory

Notice that the global sections of this sheaf are exactly the elements of $\mathbb{Q}$ since $\mathcal{O}(U_0)$ is elements integral over $\mathbb{Q}[x]$ and $\mathcal{O}(U_1)$ are elements integral over $\mathbb{Q}[1/x]$

This shows that this sheaf is not isomorphic to a structure sheaf of a ring

Indeed the global sections over a structure sheaf $Zar(R), \mathcal{O}$ form a ring isomorphic to R itself

Towards formal (Cech) cohomology

If we have a space X with a covering $X = U_0 \cup U_1$ and a sheaf $\mathcal{F}$ we can consider the map

$$\mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \rightarrow \mathcal{F}(U_0 \cap U_1)$$

$$(a_0, a_1) \longmapsto a_{1|U_0 \cap U_1} - a_{0|U_0 \cap U_1}$$

We define $H^1(U_0, U_1)$ the coker of this map

We say that $X, \mathcal{F}$ is *acyclic* iff $H^1(U_0, U_1) = 0$ for *any* covering U_0, U_1: any $b \in \mathcal{O}(U_0 \cap U_1)$ can be written of the form $a_{1|U_0 \cap U_1} - a_{0|U_0 \cap U_1}$

Towards formal cohomology

Theorem: *Any structure sheaf is acyclic*

Theorem: *If* $X = U_0 \cup U_1 = V_0 \cup V_1$ *and* U_i, V_j *are acyclic then* $H^1(U_0, U_1)$ *and* $H^1(V_0, V_1)$ *are isomorphic (as abelian group)*

Towards formal cohomology

In this way one can define the *genus* of $L = \mathbb{Q}(x, y)$ as the dimension of the $\mathbb{Q}$ vector space $H^1([x \in A], [1/x \in A]) = H^1(X, \mathcal{O})$

This is an *invariant* of L and is equal to 1 (y/x is a generator)

It does not depend on the choice of the parameter x

Theorem: *Over the field* $K = \mathbb{Q}(t)$ *we have* $H^1([t \in A], [1/t \in A]) = 0$

Corollary: *It is impossible to write* L *of the form* $\mathbb{Q}(t)$ *with* $t \in L$