Inductive Definitions

We suppose given a decidable and monotone property $G(\sigma)$ of finite sequences of binary words

"Mononotone" means $G(\sigma) \rightarrow G(\sigma w)$

We want to describe the property $B(\sigma)$ that σ is "barred" by G

$B(\sigma)$ holds if $G(\sigma)$ *or* $B(\sigma w)$ for all w

We have a logic of finite objects, but the predicate $B(\sigma)$ involves implicitely an existential quantification over a potentially infinite well-founded tree

Inductive Definitions

In general $B(\sigma)$ is not decidable

We are going to define a non-standard semantics of $B(\sigma)$ such that w.r.t. this semantics, the law of excluded-middle holds

This will show that one can assume excluded-middle for $B(\sigma)$ without having a contradiction

Inductive Definitions

Historical survey of J. Zucker in Troelstra “Mathematical Investigation of Intuitionistic Arithmetic and Analysis,” LNM 344

J. Zucker analyses ID_1^c using Gödel’s dialectica interpretation, introducing a non computable operator, the same as Hilbert’s

$$f(x) = 0 \to f(\mu(f)) = 0$$

and showing that one can find a computable majorant of all functionals defined from it, so that the introduction of μ does not matter for computing bounds

The reduction of ID_1^c to ID_1 was known before (Kreisel), but with an indirect argument via elimination of choice sequences

Before Buchholz’ introduction of the Ω rule, it was not known that ID_2^c was reducible to ID_2

General facts about Kripke models

Let M be an arbitrary poset

Theorem: The collection of downward closed subsets defines an Heyting algebra for the operations

$$F_1 \to F_2 = \{x \in M \mid \forall y \leq x . y \in F_1 \to y \in F_2\}$$

$$\wedge_i F_i = \cap F_i$$

We suppose given a distinguished downward closed subset $\bot \subseteq M$

Theorem: The collection of all subsets of the form $F \to \bot$ is a Boolean algebra H for the operations

$$\vee_i F_i = \cap_i (F_i \to \bot) \quad \to \bot \qquad \wedge_i F_i = \cap_i F_i$$

A special case: phase semantics

If M is a meet-semi lattice, and $F \subseteq M$ we define

$$F^{\perp} = \{x \in M \mid \forall y \in F.x \wedge y \in F\}$$

Lemma: $F^{\perp} = F \rightarrow \perp$ and, more generally

$$F_1 \rightarrow F_2 = \{x \in M \mid y \in F_1 \rightarrow x \wedge y \in F_2\}$$

A *fact* is a downward closed subsets of M of the form $F^{\perp}$

Thus the Boolean algebra H can be defined as the Boolean algebra of all facts

Reduction of $\mathbf{ID}_1^c$ to $\mathbf{ID}_1$

General strategy: we define a meet-semi lattice M with a subset $\bot \subseteq M$ using inductive definitions in a constructive way

M will be defined in finite terms, but the definition of $\bot$ uses an inductive definition

We build M in such a way that, relative to the corresponding Boolean algebra of facts H, we have a *model* of ID_1^c

In this way, we have "explained" ID_1^c using only ID_1

Inductive Definitions

The element of M will be finite sets of formulae of the form $B(\sigma)$, σ finite sequence of words

$\Delta = B(\sigma_1), \ldots, B(\sigma_k)$

The meet-semi lattice operation is the union

We define inductively $\vdash \Delta$.

- $\vdash \Delta, B(\sigma)$ if $G(\sigma)$ holds
- $\vdash \Delta, B(\sigma)$ if $\vdash \Delta, B(\sigma w)$ for all w

Inductive Definitions

$\vdash B(\sigma)$ means that G, classically or intuitionistically, bars σ

The meaning of $\vdash B(\sigma_1), B(\sigma_2)$ is that, classically, σ_1 or σ_2 is barred by G

This is *not the same* as $\vdash B(\sigma_1) \quad \vee \quad \vdash B(\sigma_2)$

When trying to prove

$$\vdash B(\sigma_1), B(\sigma_2) \quad \rightarrow \quad \vdash B(\sigma_1) \vee \vdash B(\sigma_2)$$

one needs

$$\forall w. F \vee G(w) \quad \rightarrow \quad F \vee \forall w. G(w)$$

which is not valid intuitionistically

Phase semantics model

We have a "phase space" $(M, \perp)$ where M is the set of finite sequents and $\perp$ the set of provable sequents

Let $X \subseteq M$ be a fact

Lemma: $\vdash \Delta, B(\sigma w)$ for all w iff $\vdash \Delta, B(\sigma)$

Corollary: $\Delta, B(\sigma w) \in X$ for all w iff $\Delta, B(\sigma) \in X$

Semantics of ID_1^c

$[\![B(\sigma)]\!] = B(\sigma)^{\perp} = \{\Delta \in M \mid \vdash B(\sigma), \Delta\}$

Lemma: We have $[\![B(\sigma)]\!] = M$ if $G(\sigma)$ holds

Lemma: We have $\cap_w [\![B(\sigma w)]\!] \subseteq [\![B(\sigma)]\!]$

Lemma: If we have a family of facts $X(\sigma)$ and $\Delta \in X(\sigma)$ whenever $G(\sigma)$ holds, and, for all σ, we have $\Delta \in (\cap_w X(\sigma w)) \rightarrow X(\sigma)$ and $\vdash \Delta, B(\sigma_0)$ then $\Delta \in X(\sigma_0)$

Corollary: B, as an H-valued predicate, satisfies the induction principle

$$(\forall \sigma . G(\sigma) \rightarrow X(\sigma)) \wedge (\forall \sigma . (\forall w . X(\sigma w)) \rightarrow X(\sigma)) \quad \rightarrow \quad B \subseteq X$$

Semantics of ID_1^c

We have $[\![B(\sigma)]\!] = M$ iff $\vdash B(\sigma)$ iff $B(\sigma)$ holds classically

We have $[\![\neg B(\sigma)]\!] = M$ iff for all Δ we have $\vdash \Delta$ whenever $\vdash \Delta, B(\sigma)$

Let H be the collection of all facts. We interpret B as an H-valued predicate $\sigma \longmapsto B(\sigma)^{\perp}$

Semantics of ID_1^c

Lemma: if $B(\sigma) \in \Delta$ then $\Delta \in [\![\neg B(\sigma)]\!]$

Lemma: $[\![\neg\neg B(\sigma)]\!] \subseteq [\![B(\sigma)]\!]$

Theorem: $\sigma \longmapsto B(\sigma)^{\perp}$ is an H-valued model of ID_1^c

Inductive Definitions

Buchholz could extend this reduction for ID_n^c to ID_n for all finite n, and even for transfinite n

However the situation is subtle for $n \geq 2$, or for the semantics of negative statements for $n = 1$: there is no known effective way to explain the classical truth of $\neg B(\sigma)$

Cut-elimination for ID_1 is achieved only for positive sequents (this is known as *partial* cut-elimination)

References

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